

GEOMETRY AND STABILITY OF NORMALIZED MEAN CURVATURE FLOW

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ABSTRACT. This paper investigates geometric stability properties and long-time behavior of a normalized mean curvature flow, where the unique smooth stationary solution is a ball of prescribed volume. We prove that the flow preserves the ρ -reflection property, which is a quantitative smallness condition, in terms of Lipschitz norms, between the set and the nearest ball. As a consequence it is shown that the evolving set becomes smooth in finite time and uniformly converges to a ball in Hausdorff topology exponentially fast. We adopt the approach developed in [19] to combine viscosity solutions approach and variational method for our analysis. The main challenge in the geometric analysis lies in the lack of comparison principle due to the normalization in the flow, which is overcome by applying a reflection comparison argument in the spirit of Serrin [30].

1. MODELS

Let Ω_0 be a bounded domain in \mathbb{R}^n with Lipschitz boundary, which is star-shaped with respect to a ball inside it (see more precise condition below). In this paper we investigate geometric properties of evolving sets $\Omega_t \subset \mathbb{R}^n$ starting with Ω_0 at $t = 0$, whose normal velocity is given by the normalizing mean curvature

$$(1.1) \quad V = -H + \lambda[|\Omega_t|]$$

where $V(x, t)$ and $H(x, t)$ respectively denote the outward normal velocity and the mean curvature at $x \in \partial\Omega_t$. The normalizing parameter $\lambda : \mathbb{R}^+ \rightarrow \mathbb{R}$ is chosen such that results in (1.1) having a unique smooth stationary solution as a ball of fixed volume s_0 (see Assumption A later in this section). An example of λ that we consider is

$$(1.2) \quad \lambda[s] = \frac{B}{s^\beta} \text{ for some } B > 0 \text{ and } \beta > 1/n.$$

Note that, for this choice of λ , due to the isoperimetric inequality (1.1) has a unique smooth stationary solution as a ball of fixed volume

$$(1.3) \quad M_0 := \left(\frac{B}{(n-1)w_n^{(1/n)}} \right)^{\frac{n}{n\beta-1}}.$$

where w_n is the volume of the unit ball in \mathbb{R}^n (See Lemma 4.2).

(1.1) can be formulated in terms of level sets, which allows us to address possible singularities of Ω_t formulated during the evolution. More precisely Ω_t can be defined by the positive set of a function $u : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$,

$$\Omega_t = \{x \in \mathbb{R}^n \mid u(x, t) > 0\},$$

where u solves the following:

$$(1.4) \quad \begin{cases} \frac{u_t}{|Du|}(x, t) = \nabla \cdot \left(\frac{Du}{|Du|} \right)(x, t) + \lambda[|\Omega_t|] & \text{for } (x, t) \in \mathbb{R}^n \times [0, \infty). \\ u(x, 0) = \chi_{\Omega_0} - \chi_{(\Omega_0)^c} & \text{for } x \in \mathbb{R}^n. \end{cases}$$

We are interested in the geometric properties of the evolving set Ω_t for non-convex initial data Ω_0 in global times. The overall difficulty in the analysis lies in possible topology changes of Ω_t . Finite-time singularities are expected even when Ω_0 has smooth boundary, due to the merging or splitting of different boundary parts in the evolution. For this reason most available literature on the global behavior of the evolution concerns convex shapes except in two dimensions(see below for more discussion on the literature).

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Possible singularities in the non-convex evolution of Ω_t necessitate a weak notion of solutions for (1.1) that allows topological changes of the sets. We will adopt two such notions developed for the mean curvature flow, the gradient flow approach originated by Almgren, Taylor, and Wang in [1], and the viscosity solutions approach through (1.4) introduced by Evans and Spruck ([15], [16], [17], and [18]). While the variational approach is useful to discuss the long-time behavior of solutions, the viscosity solutions approach is more suited to carry out geometric arguments which is necessary to show that for instance, our solution stays simply connected. Let us mention that, due to the normalizing constant λ in our problem, the standard maximum principle does not hold. For this reason our use of viscosity solutions theory must be modified from standard ones.

We will show that both variational and viscosity solutions approaches yield the same solution, which enables us to study both the long time behavior and the geometric properties of Ω_t . More precisely, we prove the following results. We assume that our initial domain satisfies ρ -reflection property (see Definition 3.2).

Theorem. *Let Ω_0 satisfy ρ -reflection property, and let λ satisfy Assumption A. Then the following holds.*

- (1) (Theorem 4.15) *There exists at least one energy solution u of (1.4) (see Definition 4.3) in the form of $u = \chi_{\Omega_t} - \chi_{(\Omega_t)^c}$ where Ω_t is a domain in \mathbb{R}^n .*
- (2) (Theorem 4.15) *u coincides with the viscosity solution w of (2.1) with the prescribed forcing term $\eta(t) := \lambda[|\Omega_t|]$ with the initial data χ_{Ω_0} .*
- (3) (Theorem 3.7) *Ω_t satisfies ρ -reflection for all times $t > 0$. In particular there exists $r_1 > 0$ such that Ω_t is star-shaped with respect to a ball B_{r_1} for all $t > 0$.*
- (4) (Lemma 5.1, Theorem 5.4) *Ω_t becomes $C^{1,1}$ in finite time. Moreover Ω_t exponentially converges to a ball of volume s_0 as $t \rightarrow \infty$ in Hausdorff distance.*

Remark 1.1. It should be pointed out that our result is not a perturbative one. ρ -reflection should be interpreted together as a quantitative smallness requirement on the Lipschitz norm distance between Ω_0 and the nearest ball (see Lemma A.1 in [19]). (4.20) yields explicit estimates on the size of the parameter ρ in terms of λ .

Due to the possible topological changes and finite-time singularities of the evolution, to study the global-in-time behavior of Ω_t it is necessary to understand the invariance of certain geometric properties under a given geometric flow. In [22], Huisken showed that initially strictly convex surfaces evolving by the mean curvature flow $V = -H$ stays convex and shrinks to a point in finite time. Parallel result is shown in [23] for the volume-preserving mean curvature flow

$$(1.5) \quad V(x, t) = -H + \lambda(t) \text{ where } \lambda(t) := \lambda[|\Omega_t|] = \frac{\int_{\partial\Omega_t} H dx}{|\partial\Omega_t|},$$

where it is shown that convex surfaces evolving by (1.5) converges into spheres. These convergence results are extended respectively to anisotropic mean curvature flow by Andrews [3], and to its volume-preserving version by Bellettini et al. in [9]. In two dimensions, for the mean curvature flow (without forcing), Angenent proves in his seminal papers [5]- [6] that the number of intersections of a pair of curves does not increase over the evolution, and in particular simply connected domain boundaries stay simply connected until it shrinks to a point.

For non-convex surfaces in dimensions larger than two, most available results address characterization of possible singularities for the mean curvature flow. Altschuler, Angenent and Giga [2] prove that rotationally symmetric hypersurfaces in \mathbb{R}^n evolving with mean curvature flow goes through singularities only at isolated points in space time. For initial surfaces with positive mean curvature (mean-convex surfaces), in [24] and [25] Huisken and Sinestrari characterized the asymptotic behavior of singularities, which was extended for star-shaped surfaces by Smoczyk [32] (also see [28]). For the mean convex surfaces Andrews proved that the surfaces preserve interior sphere conditions [4]. As for stability near the equilibrium for (1.5), in [14] Escher and Simonett uses center-manifold analysis to show that if the initial surface is sufficiently close to sphere in $C^{1,\alpha}$ sense, then the volume preserving mean curvature flow converges to the sphere.

In this paper we investigate the dynamics of “strongly star-shaped” sets evolving by (1.1). We follow the approach taken by Feldman and the first author in [19] where geometric and variational arguments to derive the long-time behavior of star-shaped droplets. The key ingredient in the proof lies in section 3, where we show the invariance of ρ -reflection property for the flow (1.1). In particular we can rule out

topological changes of Ω_t and ensure regularity property of $\partial\Omega_t$, which enables compactness argument to yield our convergence result. As for the mean curvature flows with or without forcing terms, our result appears to be one of the first results on the large time behavior of non-convex sets that are not a perturbation of the equilibrium shape.

◦ *Open questions:*

There are many questions that remains open. First of all we do not know whether general star-shaped sets stay simply connected for mean curvature type flows. We expect that our arguments, based on moving planes method, could be extended to more general settings than ρ -reflection sets, but we do not pursue this question here in this paper. The global behavior of solutions for mean-curvature flows, beyond star-shaped sets, appears to be completely open in dimensions larger than two. In terms of connection to other normalized flow, our analysis do not apply to (1.5). Apparently no direct space-time scaling applies to connect (1.1) with different λ 's, and it may be possible that different choices of λ leads to different geometric properties of solutions. Let us mention that our result would hold for (1.5) if we knew that the sets evolving with (1.5) contains a ball centered at origin with radius ρ for all times (see Lemma 3.5). Lastly, the uniqueness of (energy or any type of) solutions for (1.1) is another interesting open question.

◦ *Outline of the paper:*

Throughout the paper we assume that λ given in (1.1) satisfies the following conditions. Assumption (2) ensures that there is unique smooth stationary solution of (1.1). Assumption (3) is rather technical (see the proof of Lemma 3.5).

Assumption A.

- (1) $\lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous, locally Lipschitz function;
- (2) $\tilde{\lambda}(s) := s^{\frac{1}{n}}\lambda(s)$ is strictly decreasing, and there is a $s_0 > 0$ such that $\tilde{\lambda}(s_0) = (n-1)w_n^{\frac{1}{n}}$;
- (3) $\lambda[B_{5\rho}] > \frac{n-1}{\rho}$.

In Section 2, we discuss previous results, such as comparison principle, on viscosity solutions of mean curvature flows with forcing term,

$$(1.6) \quad V = -H + \eta(t),$$

where $\eta : \mathbb{R} \rightarrow \mathbb{R}$ is *a priori* known. In section 3 we will carry out our geometric analysis assuming that there exists a viscosity solution of (1.6) with $\eta(t)$ coinciding with $\lambda[|\Omega_t|]$. The existence of such solutions will be shown later in section 4 by a coincidence theorem, Theorem 4.15, which identifies an energy solution constructed by variational approach as a viscosity solution with the desired $\eta(t)$.

In section 3 we show that when the initial set Ω_0 satisfies ρ -reflection, Ω_t evolving by (1.4) As in [19] our geometric arguments are based on the *moving planes method*, based on reflection comparison principle of (1.1). More precisely, let $H_\nu^+ := \{x + x_0 : x \cdot \nu \geq 0\}$ and $H = H_\nu(x_0) := \partial H_\nu^+$. Since the normal velocity law (1.1) is preserved with respect to spatial reflections, the parabolic comparison principle applies in the region $H_\nu^+ \times [0, \infty)$ to Ω_t and $\tilde{\Omega}_t^{\nu, x_0}$, the reflected version of Ω_t with respect to H_ν . It follows that if

$$(1.7) \quad \tilde{\Omega}_0^{\nu, x_0} \subset \Omega_0 \text{ in } H_\nu^+,$$

then such property is preserved for later times. This property implies that the solution χ_{Ω_t} is increasing with respect to ν -direction on H . Hence by considering a family of hyperplanes H going through x_0 with varying normal directions which satisfies (1.7), we are able to show that the evolving interface $\partial\Omega_t$ stays locally Lipschitz in space *at the time when* it passes through x_0 . Serrin [30] used a similar argument to show that the only classical stationary solutions of (1.1) are radial. Let us emphasize again that the dynamic problem (1.1) does not satisfy classical comparison principle between solutions with different initial data, due to the nonlocal dependence of λ on the solution. This is the main difficulty in our geometric analysis, and this is also why we resort exclusively to this particular type of comparison argument.

In section 4, based on the discrete-time gradient flow scheme originated by Almgren-Taylor Wang [1], we generate an energy solution of (1.4) as an evolving family of sets $(\Omega_t)_{\{t>0\}}$, characterized as the

continuum gradient flow of the set-dependent energy functional

$$(1.8) \quad J(E) = \text{Per}(E) - \Lambda[|E|], \quad \Lambda'(s) = \lambda(s).$$

In Theorem 4.6, we show that the discrete-time approximation of the energy solution satisfies a barrier property, following the argument of [21]. Using this it follows Theorem 4.15 that the energy solution is a viscosity solution of (1.6) with $\eta(t) = \lambda[|\Omega_t|]$. Now the geometric analysis developed in section 3 applies to our energy solution. Let us mention that, as in [19], we need to consider the gradient flows with geometric restrictions (see Definition 4.1 and (4.4)) to ensure pointwise properties of the energy solution. This requires rather subtle modification of the analysis carried out in [21].

In section 5 we investigate the large-time behavior of star-shaped sets Ω_t . Note that the unique (up to translation) stationary point of $J(E)$, among star-shaped sets, is a ball B of volume s_0 by Assumption A. Using this fact and the gradient flow structure of (1.4), we obtain that Ω_t converges to a ball with volume s_0 in Hausdorff distance (Theorem 5.2). Furthermore we are able to show that the convergence is in almost- $C^{1,1}$ sense (Lemma 5.3). Such regularity result invokes the center manifold approach taken in [14] to yield the exponential convergence of Ω_t to a unique ball B with volume s_0 as $t \rightarrow \infty$.

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2. NOTATIONS AND PROPERTIES OF VISCOSITY SOLUTIONS

In this section, we recall viscosity solutions theory for the mean curvature flow with a priori given forcing term, as well as its connection to fronts moving with the normal velocity (1.6). We begin with a couple of definitions.

- For a function $h : Q := \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ we denote its positive set by

$$\Omega_t(h) := \{x : h(x, t) > 0\},$$

- For a function $f : A \in Q \rightarrow \mathbb{R}$ we denote its lower and upper semi-continuous envelopes by

$$f_*(x) = \lim_{\epsilon \downarrow 0} \inf_{|x-y|+|t-s|<\epsilon, (y,s) \in Q} f(y)$$

and

$$f^*(x) = \lim_{\epsilon \downarrow 0} \sup_{|x-y|+|t-s|<\epsilon, (y,s) \in Q} f(y).$$

For a given continuous function $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}$, let us consider the level set PDE of (1.6) in $\mathbb{R}^n \times [0, \infty)$:

$$(2.1) \quad \frac{u_t}{|Du|}(x, t) = \nabla \cdot \left(\frac{Du}{|Du|} \right)(x, t) + \eta(t)$$

with initial data

$$u(x, 0) = u_0(x) := \chi_{\Omega_0} - \chi_{(\overline{\Omega_0})^c} \text{ for } x \in \mathbb{R}^n.$$

For later purposes we also consider viscosity solutions for the *restricted* flow with a constant $M > \|\eta\|_\infty$,

$$(2.2) \quad \frac{u_t}{|Du|}(x, t) = \max \left\{ \nabla \cdot \left(\frac{Du}{|Du|} \right) + \eta(t), -M \right\}(x, t).$$

Definition 2.1. ([12, Definition 2.1]) [7, Definition 6.1])

- (a) A function $u : Q \rightarrow \mathbb{R}$ is a *viscosity subsolution* of (2.1) if upper semicontinuous relaxation $u^* < \infty$ and for each pair $\phi \in C^{2,1}(Q)$ and $(x_0, t_0) \in Q$ satisfying that $u^* - \phi$ has a local maximum at (x_0, t_0) , it holds that

$$\frac{\phi_t}{|D\phi|}(x_0, t_0) \leq \nabla \cdot \left(\frac{D\phi}{|D\phi|} \right)(x_0, t_0) + \eta(t_0).$$

- (b) A function $u : Q \rightarrow \mathbb{R}$ is a *viscosity supersolution* of (2.1) if lower semicontinuous relaxation $u_* < \infty$ and for each pair $\phi \in C^{2,1}(Q)$ and $(x_0, t_0) \in Q$ satisfying that $u_* - \phi$ has a local minimum at (x_0, t_0) , it holds that

$$\frac{\phi_t}{|D\phi|}(x_0, t_0) \geq \nabla \cdot \left(\frac{D\phi}{|D\phi|} \right)(x_0, t_0) + \eta(t_0).$$

- (c) A function $u : Q \rightarrow \mathbb{R}$ is a *viscosity solution* of (2.1) with initial data $u_0(x)$ if u^* is a *viscosity subsolution* and u_* is a *viscosity supersolution*, and if $u^* = (u_0)^*$ and $u_* = (u_0)_*$ at $t = 0$.

Parallel definitions can be made for viscosity sub- and supersolutions of (2.2). The following lemma follows from above definition: we refer to [13] for further discussion on the stability properties of viscosity solutions.

- Lemma 2.1.** (a) For $n \in \mathbb{N}$, let $u_n := \chi_{\Omega_t^n} - \chi_{(\Omega_t^n)^c}$ be a viscosity solution of (2.2) in Q . If $\partial\Omega_t^n$ converges to $\partial\Omega_t$ as $n \rightarrow \infty$ in Hausdorff distance, uniformly for all $t > 0$, then $u := \chi_{\Omega_t} - \chi_{(\Omega_t)^c}$ is a viscosity solution of (2.1).
- (b) For $n \in \mathbb{N}$, let u_n as given above be a viscosity solution of (2.2) in Q with $M = M_n$ and with initial data u_0 . If $\partial\Omega_t^n$ uniformly converges to $\partial\Omega_t$ in Hausdorff distance, uniformly for all $t > 0$, then u is a viscosity solution of (2.1).

Existence, uniqueness and stability properties for viscosity solutions of (2.1)-(2.2) follows from [12].

Theorem 2.2. [12, Theorem 4.1, 6.7, 6.8]

- (1) Let u and v be a subsolution and supersolution, respectively, of (2.1) (or (2.2)) For the bounded domain Ω , if $u^* \leq v_*$ on $\partial_p\Omega_T = \{0\} \times \Omega \cup [0, T] \times \partial\Omega$, then we have

$$u^* \leq v_* \text{ on } \Omega_T := (0, T] \times \Omega.$$

- (2) For given bounded domain $\Omega_0 \subset \mathbb{R}^n$ with Lipschitz boundary, there exists a unique viscosity solution of (2.1) (or (2.2)) with initial data $u_0(x)$.

From Theorem 4.2.1 in [20], for any Ω_0 , the unique viscosity solution of (2.1) is of the form

$$\tilde{u}(x, t) = \chi_{\Omega_t(u)} - \chi_{(\overline{\Omega}_t(u))^c}.$$

is the unique viscosity solution of (2.1), where Ω_t satisfies (1.6) if $\partial\Omega_t$ is C^2 . We will thus consider the set Ω_t obtained from the above viscosity solutions formulation as a weak notion of sets evolving by (1.6).

Remark 2.3. Let us point out that viscosity solutions theory is built on the comparison principle property (Theorem 2.1 (1)) of the problem. For our original problem (1.1)-(1.2) it is easy to see that such property fails to hold from testing radial profiles. Indeed the well-posedness of (1.1)-(1.2) will be established later in section 4 where we show the existence of viscosity solutions for (1.6) where $\eta(t)$ coincides with $\lambda[|\Omega_t|]$ given by (1.2) (see Defintion 2.2).

As in [19], we can define the solution of (1.1)-(1.2) below.

Definition 2.2. A function $u : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ is a *subsolution* (*supersolution*) of (1.4) if u is a viscosity solution (supersolution) of (2.1) for the forcing term $\eta(t) = \lambda[|\Omega_t(u)|]$. A function u is a *solution* of (1.4) with initial positive phase Ω_0 if u is a viscosity solution of (1.4) with initial positive phase Ω_0 .

Next we introduce a regularization procedure that is by now well-known for free boundary problems (see [10] and [26]). These regularizations are useful in our geometric analysis in section 4 (see the proof of Lemma 4.12).

Lemma 2.4. (Convolutions in space) Let u be a viscosity supersolution of (2.1). Then, there exists a_0 such that for all $a \in (0, a_0)$,

$$\tilde{u}(x, t) := \inf_{y \in \overline{B}_{a-ct}(x)} u(y, t),$$

is a viscosity supersolution of

$$(2.3) \quad \begin{cases} \frac{u_t}{|Du|}(x, t) = \nabla \cdot \left(\frac{Du}{|Du|} \right)(x, t) + \eta(t) + c & \text{for } (x, t) \in \mathbb{R}^n \times [0, \infty), \\ u(x, 0) = \tilde{g}(x) = \inf_{y \in \overline{B}_a(x)} g(y) & \text{for } x \in \mathbb{R}^n, \end{cases}$$

for $x \in \mathbb{R}^n$ and $0 \leq t < \frac{a}{c}$.

Similarly, let u be a viscosity subsolution of (2.1). Then, there exists a_0 such that for all $a \in (0, a_0)$,

$$\tilde{u}(x, t) := \sup_{y \in \overline{B}_{a-ct}(x)} u(y, t)$$

is a viscosity subsolution of

$$\begin{cases} \frac{u_t}{|Du|}(x, t) = \nabla \cdot \left(\frac{Du}{|Du|} \right)(x, t) + \eta(t) - c & \text{for } (x, t) \in \mathbb{R}^n \times [0, \infty) \\ u(x, 0) = \tilde{g}(x) = \sup_{y \in \overline{B}_a(x)} g(y) & \text{for } x \in \mathbb{R}^n \end{cases}$$

for $x \in \mathbb{R}^n$ and $0 \leq t < \frac{a}{c}$.

Proof. First, there exists $a_0 > 0$ such that new initial data \tilde{g} of \tilde{u} is equal to the characteristic function $\tilde{\Omega}_0 \subset \mathbb{R}^n$,

$$\tilde{g}(x) = \inf_{y \in \tilde{B}_a(x)} g(y) = \chi_{\tilde{\Omega}_0} - \chi_{(\tilde{\Omega}_0)^c}.$$

Then, for all $a \in (0, a_0)$, \tilde{g} is well-defined by a characteristic function.

Next, let us show that \tilde{u} is a viscosity supersolution of (2.3). Suppose that for any function $\phi \in C^2(\mathbb{R}^n \times [0, \frac{a}{c}))$, $\tilde{u} - \phi$ has a local minimum at $(x_0, t_0) \in \mathbb{R}^n \times [0, \frac{a}{c})$. By subtracting the minimum value, we can assume that

$$(\tilde{u} - \phi)(x_0, t_0) = 0.$$

Moreover, since \tilde{u} is a characteristic function, we can assume that $\tilde{u}(x_0, t_0) = \phi(x_0, t_0) = 0$

By definition of \tilde{u} , there exists $x_1 \in \mathbb{R}^n$ such that

$$|x_1 - x_0| \leq a - ct_0 \text{ and } \tilde{u}(x_0, t_0) = u(x_1, t_0).$$

By construction of \tilde{u} , a ball $B_{a-ct_0}(x_1)$ is in $\Omega_{t_0}(\tilde{u})^c$. Since the point (x_0, t_0) is on the boundary of $\Omega_{t_0}(\tilde{u})$, we conclude that $|x_1 - x_0| = a - ct_0$.

Now, let us define new test function ψ by,

$$\psi(x, t) = \phi(x + (a - ct)\vec{n}, t),$$

where the outward normal vector of $\partial\Omega_{t_0}(\tilde{u})$ at x_0 is equal to

$$\vec{n} = \frac{x_1 - x_0}{|x_1 - x_0|}.$$

Let us show that $u - \phi$ has a local minimum at (x_1, t_0) . First, there exists $\delta > 0$ such that for all $(x, t) \in B_\delta(x_0, t_0)$, we have

$$\begin{aligned} \tilde{u}(\tilde{x}, t) - \phi(\tilde{x}, t) &\geq \tilde{u}(x_0, t_0) - \phi(x_0, t_0), \\ &\geq u(x_1, t_0) - \psi(x_1, t_0). \end{aligned}$$

Then, for $(\tilde{x}, t) \in B_\delta(x_1, t_0)$, we have

$$\begin{aligned} u(\tilde{x}, t) - \psi(\tilde{x}, t) &\geq \inf_{y \in B_{a-ct}(\tilde{x} + (a-ct)\vec{n})} \{u(y, t)\} - \phi(\tilde{x} + (a-ct)\vec{n}, t), \\ &= \tilde{u}(\tilde{x} + (a-ct)\vec{n}, t) - \phi(\tilde{x} + (a-ct)\vec{n}, t), \\ &\geq \tilde{u}(x_0, t_0) - \phi(x_0, t_0) = u(x_1, t_0) - \psi(x_1, t_0). \end{aligned}$$

The second inequality follows from $(\tilde{x} + (a-ct)\vec{n}, t) = (\tilde{x} - x_1 + x_0, t) \in B_\delta(x_0, t_0)$.

Therefore, we could get the following inequality:

$$\frac{\psi_t}{|D\psi|}(x_1, t_0) \geq \nabla \cdot \left(\frac{D\psi}{|D\psi|} \right)(x_1, t_0) + \eta(t),$$

As the level set,

$$\{x \in \mathbb{R}^n \mid \phi(x, t_0) = \phi(x_0, t_0)\},$$

touches $\partial\Omega_t(\tilde{u})$ at (x_0, t_0) , so \vec{n} is equal to the outward normal vector $-\frac{D\phi}{|D\phi|}$ of the level set of ϕ , we have

$$\frac{\phi_t}{|D\phi|}(x_0, t_0) + c \left(\frac{D\phi \cdot \vec{n}}{|D\phi|}(x_0, t_0) \right) \geq \nabla \cdot \left(\frac{D\phi}{|D\phi|} \right)(x_0, t_0) + \eta(t),$$

which is equivalent to

$$\frac{\phi_t}{|D\phi|}(x_0, t_0) \geq \nabla \cdot \left(\frac{D\phi}{|D\phi|} \right)(x_0, t_0) + c + \eta(t).$$

Therefore, \tilde{u} is viscosity supersolution of

$$\frac{u_t}{|Du|}(x, t) = \nabla \cdot \left(\frac{Du}{|Du|} \right)(x, t) + \eta(t) + c.$$

Similarly, we can show that sup-convolution become subsolution of

$$\frac{u_t}{|Du|}(x, t) = \nabla \cdot \left(\frac{Du}{|Du|} \right)(x, t) + \eta(t) - c,$$

by taking the new test function defined by

$$\psi(x, t) = \phi(x - (a - ct)\vec{n}, t).$$

□

3. PRESERVATION OF STARSHAPEDNESS

In this section, we show that the mean curvature flow with the volume dependent forcing term (1.4) is star-shaped with respect to a ball for all time. In [19], Feldman and Kim introduced a strong version of star-shapedness, called ρ -reflection, to study long time behavior of dynamic capillary drops. We introduce these concepts to our problem and discuss stability properties of (1.4) with respect to ρ -reflection.

We first consider a solution u of normalized mean curvature flow with a priori given forcing term (2.1) discussed in section 2. Based on the moving planes method, the model (2.1) has reflection comparison principle (Theorem 3.3). This property will yield that $\Omega_t(u)$ will satisfy the ρ -reflection property as long as $\Omega_t(u)$ contains the closure of the ball B_ρ . Next we consider the solution of our original equation (1.4) defined in Definition 2.2, assuming that such solution exists. We show that $\Omega_t(u)$ contains $\overline{B_\rho}$ for all times due to its normalization property. Therefore we conclude that for our problem $\Omega_t(u)$ satisfies ρ -reflection, and thus is star-shaped with respect to B_ρ for all time. The existence of such solution will be discussed in the next section.

Let us begin with several geometric definitions.

Definition 3.1. A bounded set Ω in \mathbb{R}^n is *star-shaped with respect to a ball* B_r if for any point $y \in B_r$, Ω is star-shaped with respect to y . Let

$$S_r := \{\Omega : \text{star-shaped with respect to } B_r(0)\} \text{ and } S_{r,R} := S_r \cap \{\Omega : \Omega \subset B_R(0)\}.$$

Let us define the reflection with respect to the hyperplane H by

$$(3.1) \quad \phi_H(x) = x - 2\langle x - y, \nu \rangle \nu$$

where ν is the unit normal vector of H and y can be chosen any vector in H .

Moreover, let us denote a hyperplane which is perpendicular to a unit vector $\nu \in \mathbb{R}^n$ and passes through zero by H_ν . Also, let us denote the parallel translation of H_ν in ν direction by

$$H_\nu(s) = H_\nu + s\nu$$

and the half plane divided by hyperplane $H_\nu(s)$ containing zero by $H_{\nu,-}(s)$ and the opposite half plane by $H_{\nu,+}(s)$ for $s > 0$ that is

$$H_{\nu,+}(s) := \{x \in \mathbb{R}^n : x \cdot \nu > s\} \text{ and } H_{\nu,-}(s) := \{x \in \mathbb{R}^n : x \cdot \nu < s\}.$$

Definition 3.2. A bounded, open set Ω satisfies ρ -reflection if

- (i) Ω contains $\overline{B_\rho(0)}$ and
- (ii) Ω satisfies that

$$\phi_{H_\nu(s)}(\Omega \cap H_{\nu,+}(s)) \subset \Omega \cap H_{\nu,-}(s)$$

for all direction $\nu \in S^{n-1}$ and all $s > \rho$.

The ρ -reflection can be viewed as a smallness condition on the Lipschitz norm distance between $\partial\Omega$ and the nearest ball (see the Appendix in [19].) The following lemma states several properties and the relationship between the two concepts introduced above, ρ -reflection and S_r . In particular we can see that the first condition in the definition of ρ -reflection, $\overline{B_\rho(0)} \subset \Omega$ is necessary in order to make the set Ω star-shaped with respect to a ball.

Lemma 3.1. [19, Lemma 3, 9, 10]

- (1) For a bounded domain Ω containing $B_r(0)$, the followings are equivalent
 - (i) Ω is in S_r .
 - (ii) There exists $\epsilon_0 > 0$ such that

$$\Omega \subset \bigcap_{|z| \leq a\epsilon} [(1+\epsilon)\Omega + z]$$

for all $0 < \epsilon < \epsilon_0$ and $0 < a < r$.

- (iii) For all $x \in \partial\Omega$,

$$(x + C(-x, \theta_x)) \cap C(x, \frac{\pi}{2} - \theta_x) \cup B_r(0) \subset \Omega$$

where

$$\sin \theta_x = \frac{r}{|x|} \text{ and } C(x, \theta) = \{y \mid \langle x, y \rangle \geq \cos \theta |x||y|\}.$$

(iv) There exists $\epsilon > 0$ such that for all $x \in \partial\Omega$

$$x + C(-x, \theta_x) \cap B_\epsilon(x) \subset \Omega^c \text{ where } \sin \theta_x = \frac{r}{|x|}.$$

(2) Suppose that Ω satisfies ρ -reflection. Then, we have

$$\sup_{x \in \partial\Omega} |x| - \inf_{x \in \partial\Omega} |x| \leq 4\rho.$$

(3) Suppose that Ω satisfies ρ -reflection. Then, Ω is in S_r where r is given by

$$r = \left(\inf_{x \in \partial\Omega} |x|^2 - \rho^2 \right)^{1/2}.$$

The following lemma is due to the interior and exterior cone property of *star-shapedness with respect to a ball* B_r .

Lemma 3.2. For a smooth function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$, assume that ϕ has bounded and open positive set,

$$\Omega = \{x \in \mathbb{R}^n \mid \phi(x) > 0\},$$

which has a compact hypersurface $\partial\Omega$. Then, the followings are equivalent:

(1) Ω is in S_r .

(2) For all $x \in \partial\Omega$, we have

$$x \cdot \vec{n}_x = x \cdot \left(-\frac{D\phi}{|D\phi|}(x) \right) \geq r$$

where \vec{n}_x is the outward unit normal of $\partial\Omega$ at x .

Moreover, strong comparison principle for both bounded and unbounded set Theorem.2.2 implies reflection comparison principle.

Theorem 3.3. (Reflection Comparison) Let u be a viscosity solution of (2.1). Let H be a hyperplane \mathbb{R}^n such that $H \cap B_\rho(0) = \emptyset$ and define H_+ and H_- the half spaces divided by H satisfying that H_+ contains $B_\rho(0)$. Then, for reflection ϕ_H , (3.1), with respect to H , $u(\phi_H(x), t)$ is a viscosity solution in $(x, t) \in H_+ \times [0, \infty)$. Moreover, if we have

$$u(\phi_H(x), 0) \leq u(x, 0)$$

for all $x \in H_+$ then we have

$$u(\phi_H(x), t) \leq u(x, t)$$

for all $x \in H_+$ and $t > 0$.

Under the assumption that $\Omega_0(u)$ satisfies ρ -reflection, we have the same relation

$$u(\phi_H(x), t) \leq u(x, t)$$

for all $x \in H_+$ and $t > 0$.

Therefore, we can conclude that $\Omega_t(u)$ satisfies ρ -reflection if $B_\rho \subset \Omega_t(u)$ for $t \in [0, \infty)$.

Theorem 3.4. Let u be a viscosity solution of (2.1). In addition assume that $\Omega_0(u)$ satisfies ρ -reflection. Let $I = [0, T)$ be the maximal interval satisfying $\overline{B_\rho} \subset \Omega_t(u)$. Then, $\Omega_t(u)$ satisfies ρ -reflection, and therefore star-shaped for $t \in I$.

Proof. Let us show that $\Omega_t(u)$ satisfies ρ -reflection for all $t \in I$. From the definition of ρ -reflection, it is enough to show that

$$(3.2) \quad \phi_{H_\nu(\rho)}(\Omega_t \cap H_{\nu,+}(\rho)) \subset \Omega_t \cap H_{\nu,-}(\rho)$$

for all unit vector $\nu \in \mathbb{R}^n$.

Since Ω_0 satisfies ρ -reflection, we have

$$\phi_{H_\nu(\rho)}(\Omega_0 \cap H_{\nu,+}(\rho)) \subset \Omega_0 \cap H_{\nu,-}(\rho).$$

The fact that u is a characteristic function implies

$$u(\phi_{H_{\nu,+}(\rho)}(x), 0) \leq u(x, 0)$$

for $x \in H_{\nu,+}(\rho)$. Then, by reflection comparison Theorem 3.3, we have

$$u(\phi_{H_{\nu,+}(\rho)}(x), t) \leq u(x, t)$$

for $x \in H_{\nu,+}(\rho)$, which is equivalent to (3.2). □

In contrast to Theorem 2.2 and Theorem 3.3, in the above theorem we only consider the one fixed flow. Therefore it follows that, if u is solution of (1.4) in the sense of Definition 2.2, $\Omega_t(u)$ is star-shaped as long as $\overline{B_\rho} \subset \Omega_t(u)$.

Lemma 3.5. *Suppose that there exists a solution u of (1.4) in the sense of Definition 2.2. In addition assume that $\Omega_0(u)$ satisfies ρ -reflection. Then, there exists $a > 0$ such that $B_{(1+a)\rho} \subset \Omega_t(u)$ for all $t > 0$ where constant a only depends on $\Omega_0(u)$.*

Proof. By continuity of the function λ and Assumption A(3), there exists $a > 0$ such that

$$(3.3) \quad B_{(1+a)\rho} \subset \subset \Omega_0(u) \text{ and } \lambda[B_{(5+a)\rho}] > H[B_{(1+a)\rho}].$$

Suppose that there exists $t_* > 0$ such that $B_{(1+a)\rho}$ is not contained in $\Omega_{t_*}(u)$. Then, there exists (x_0, t_0) such that $\Omega_t(u)$ first touches below $B_{(1+a)\rho}$ at (x_0, t_0) . Then, by Lemma 3.1, we have

$$\sup_{x \in \partial\Omega_{t_0}(u)} |x| \leq [4\rho + \inf_{x \in \partial\Omega_{t_0}(u)} |x|] = (5+a)\rho,$$

and thus $\Omega_{t_0}(u)$ is contained in $B_{(5+a)\rho}$, and we have

$$(3.4) \quad |\Omega_{t_0}(u)| \leq |B_{(5+a)\rho}|$$

From Assumption A(2), λ is a multiplication of two strictly decreasing functions, $\tilde{\lambda}$ and $\frac{1}{x^{\frac{1}{n}}}$, so λ is a strictly decreasing function. By (3.3) and (3.4), we get

$$\lambda[|\Omega_{t_0}(u)|] \geq \lambda[|B_{(5+a)\rho}|] > H[B_{(1+a)\rho}].$$

Let us define function $\phi : \mathbb{R}^n \times [0, \infty)$ by

$$\phi(x, t) := ((1+a)\rho)^2 - |x|^2.$$

Then, we have

$$\frac{\phi_t}{|D\phi|}(x_0, t_0) = 0 < -H[B_{(1+a)\rho}] + \lambda[|\Omega_{t_0}(u)|] = \nabla \cdot \left(\frac{D\phi}{|D\phi|} \right)(x_0, t_0) + \lambda[|\Omega_{t_0}(u)|],$$

so ϕ is a strictly subsolution in the small neighborhood of t_0 . However, by the comparison principle (Theorem 2.2) for $\tilde{u} := \chi_{\Omega_t(u)}$ and ϕ , this contradicts the fact that $\Omega_{t_0}(u)$ first touches $B_{(1+a)\rho}$ at x_0 . \square

Lemma 3.6. *Suppose that there exists a solution u of (1.4) in the sense of Definition 2.2. In addition assume that $\Omega_0(u)$ satisfies ρ -reflection. Then, there exists $R > 0$ such that $\Omega_t(u) \subset B_R$ for all $t > 0$.*

Proof. Due to Assumption A(2), there exists s_0 such that $s_0^{\frac{1}{n}} \lambda(s_0) = (n-1)w_n^{\frac{1}{n}}$. Thus, we get

$$|B_{r_*}|^{\frac{1}{n}} \lambda(|B_{r_*}|) < (n-1)w_n^{\frac{1}{n}}$$

for any $r_* > 0$ such that $|B_{r_*}| > s_0$. For sufficiently large $R > r_*$ such that

$$\Omega_0(u) \subset \subset B_R, \quad |B_{R-4\rho}| > s_0 \quad \text{and} \quad \frac{4\rho(n-1)}{R} < \epsilon := (n-1) - r_* \lambda(|B_{r_*}|),$$

let us assume that $\Omega_t(u)$ first touches B_R at (x_0, t_0) . Then, by the same argument in the proof of Lemma 3.5, we have

$$\lambda[|\Omega_t(u)|] \leq \lambda[|B_{R-4\rho}|] \leq \frac{n-1-\epsilon}{R-4\rho} < \frac{(n-1)(1-\frac{4\rho}{R})}{R} = \frac{n-1}{R}.$$

Since at the touching point x_0 , the curvature of Ω_t is greater than the curvature of B_R , $\frac{n-1}{R}$, this above inequality implies that the outward normal velocity is negative. This contradicts that Ω_t first touches B_R at (x_0, t_0) . \square

By combining with Lemma 3.4 and 3.5, we can conclude the following theorem.

Theorem 3.7. *Suppose that there exists a solution u of (1.4) in the sense of Definition 2.2. In addition assume that $\Omega_0(u)$ satisfies ρ -reflection. Then, $\Omega_t(u)$ also satisfies ρ -reflection. Moreover, there exists $r_1 > 0$ such that $\Omega_t(u)$ is in S_{r_1} for all $t > 0$ where r_1 only depends on positive set of the initial data Ω_0 and ρ .*

Proof. By the above Lemma 3.5, there exists $a > 0$ such that $B_{(1+a)\rho} \subset \Omega_t(u)$ for all $t > 0$.

Let us fix $\eta(t) = \lambda[|\Omega_t(u)|]$ for the given u . Then, we can apply Lemma 3.4. for $u(x, t)$ and $\eta(t)$, and $\Omega_t(u)$ satisfies ρ -reflection. Finally, by Lemma 3.1, $\Omega_t(u)$ is in S_r for

$$r = \left(\inf_{x \in \partial\Omega} |x|^2 - \rho^2 \right)^{1/2} \geq r_1 := a^2 \rho^2 + 2a\rho.$$

Therefore ρ , and $\Omega_t(u)$ is *star-shaped with respect to a ball* B_{r_1} for all $t > 0$. \square

Note that if Ω_0 satisfies ρ -reflection, then for small $\varepsilon > 0$ the sets $\Omega_0^{\varepsilon,+} := (1 + \varepsilon)\Omega_0$ and $\Omega_0^{\varepsilon,-} := (1 + \varepsilon)^{-1}\Omega_0$ satisfies $(1 + \varepsilon)\rho$ -reflection. Therefore we have the following corollary, which will be used in section 4.

Corollary 3.8. *If Ω_0 satisfies ρ -reflection, then for sufficiently small $\varepsilon > 0$ the viscosity solutions starting from $\Omega_0^{\varepsilon,\pm}$ has their positive sets in S_r for all $t > 0$, where $r = r_1 - O(\varepsilon)$.*

Before we proceed to the next section, we show that $\Omega_t(u)$ Hölder continuous in time.

Lemma 3.9. *Let u be a viscosity solution of (2.1) or (2.2). For $K := \|\eta\|_\infty$, let $x_0 \in \partial\Omega_t(u)$ and suppose $B_{2cr}(x_0 + r\nu)$ lies outside of $\Omega_t(u)$ for a unit vector ν and some constant c and $r > 0$ satisfying $cr \leq \frac{1}{K}$. Then, $B_{cr}(x_0 + r\nu)$ lies outside of Ω_s for $t \leq s \leq t + \frac{(cr)^2}{n}$.*

Proof. Let us compare $\Omega_t(u)$ with the radial subsolution $B_{r(s)}(x_0 + r\nu)$, where $r(t) = 2cr$ and $r'(s) = -\frac{n-1}{cr} - K$. Suppose that $\Omega_t(u)$ touches $B_{r(s)}(x_0 + r\nu)$ at (x_0, t_0) before $t_0 < t + \frac{(cr)^2}{n}$. Then, at (x_0, t_0) , the normal velocity of Ω_{t_0} bounded by the curvature of $B_{r(s)}(x_0 + r\nu)$, so we have

$$V_1 \leq -H_1 + \eta(t) \leq H_2 + K$$

where V_1 and V_2 are the outward normal velocity of Ω_{t_0} and $B_{r(s)}(x_0 + r\nu)$ at x_0 , and H_1 and H_2 are the mean curvature of Ω_{t_0} and $B_{r(s)}(x_0 + r\nu)$ at x_0 , respectively.

On the other hand, since we have

$$t_0 < t + \frac{(cr)^2}{n} < t + \frac{(cr)^2}{(n-1) + K(cr)},$$

the radius of the ball is great than cr at t_0 . So, we have

$$H_2 + K < \frac{n-1}{cr} + K = -V_2 \text{ and } V_1 + V_2 < 0.$$

This contradicts to the fact that Ω_t touches $B_{r(s)}(x_0 + r\nu)$ at (x_0, t_0) . \square

Since $\Omega_t(u)$ is locally Lipschitz as Lemma 3.1, we can apply above lemma for any $r > 0$ with uniform constant $c > 0$. We conclude that the boundary moves with Hölder continuity over time:

Corollary 3.10. *Let u be a viscosity solution of (2.1) or (2.2). Then there exists C independent of M such that we have*

$$d_H(\Omega_t(u), \Omega_s(u)) \leq C|s - t|^{1/2} \text{ for } s, t \geq 0.$$

4. EXISTENCE OF THE FLOW: COINCIDENCE BETWEEN ENERGY AND VISCOSITY SOLUTIONS.

In this section, we show the existence of the initial value problem of normalized mean curvature flow (1.4) under the assumption that positive set of the initial data satisfies ρ -reflection. Our approach is based on the discrete-time gradient flow first introduced by Almgren-Taylor-Wang [1], which is very useful when proving existence of weak solutions for evolution problems with underlying variational structure (see for instance [29], [11], [9] which all addresses mean-curvature type flows).

Our goal in this section is to show the existence of a solution of (1.4) in the sense of Definition 2.2, by using the variational tools. As a result our solution obtained this way also dissipates energy, which will be used in section 5 to describe its long time behavior. We start by introducing the notion of *energy solution* based on the gradient flow structure of the problem (see Definition 4.2). We will show that this energy solution (let's call it w) coincides with the viscosity solution of (2.1) with fixed multiplier $\eta(t) := \lambda(|\Omega_t(w)|)$, which then characterizes w as a solution of (1.4).

Showing such result is unfortunately a bit involved, due to additional difficulties we describe below. As in [19], we introduce a gradient flow with geometric constraint (given as $A_{r_0, M}$ below). We point

out that our geometric constraint is weaker than the one we obtained in Theorem 3.7, to ensure that the constraint is not artificial. Our constraint is crucial to ensure the strong (in Hausdorff distance) convergence of the discrete gradient flow, which enables geometric analysis of the limiting flow. On the other hand the constraint also poses technical challenges when we compare the energy solution with the viscosity solutions. This is why we first approximate our original problem with a “restricted” version (4.3).

Let us give a brief summary of this section. We begin with showing that (restricted) energy solutions have a “constrained” barrier property (Lemma 4.6) with respect to classical solutions (Lemma 4.6). Based on this barrier property, we can show that the energy solution coincides with the corresponding viscosity solution as long as the viscosity solution is star-shaped (Lemma 4.11). Ensuring this star-shaped property for the viscosity solution is the last step leading to the coincidence result (Lemma 4.13): this is where we need the lower bound on the velocity of the flow imposed by (4.3). After we show the coincidence between the restricted energy solution and the corresponding viscosity solution, we address removing the bound M in (4.3) to obtain our desired result.

4.1. Restricted gradient flow. The energy corresponding to the classical mean curvature flow, where $\lambda \equiv 0$, is

$$(4.1) \quad \text{Per}(E) = \inf \left\{ \left| \int_E \nabla \cdot \phi \right| \mid \phi \in C_c(E; \mathbb{R}^n), |\phi| \leq 1 \right\} \text{ for } E \in \mathbb{R}^n.$$

As our model (1.4) has the additional forcing term λ , the corresponding energy is equal to

$$(4.2) \quad J(E) = \text{Per}(E) - \Lambda[|E|],$$

where the function $\Lambda(s)$ is an anti-derivative of $\lambda(s)$.

For the sets E and F in \mathbb{R}^n , we use the pseudo-distance defined by

$$\tilde{d}(E, F) = \left(\int_{E \Delta F} d(x, \partial E) dx \right)^{\frac{1}{2}}$$

where $E \Delta F = (E \setminus F) \cup (F \setminus E)$.

For technical reasons (see Lemma 4.13), we first consider the following equation:

$$(4.3) \quad \begin{cases} \frac{u_t}{|Du|}(x, t) = \max \left\{ \nabla \cdot \left(\frac{Du}{|Du|} \right) (x, t) + \lambda[|\Omega_t(u)|], -M \right\} & \text{for } (x, t) \in \mathbb{R}^n \times [0, \infty), \\ u(x, 0) = g(x) := \chi_{\Omega_0} - \chi_{(\overline{\Omega_0})^c} & \text{for } x \in \mathbb{R}^n. \end{cases}$$

Let r_1 the constant given in Lemma 3.5, which depends only on Ω_0 , and let S_r be as defined in Definition 3.1. For a fixed constant $r_0 < r_1$, we define the one-step discrete gradient flow $T_{h,M}$ of the energy J with admissible class

$$(4.4) \quad A_{r_0,M}(E) := \{F \in S_{r_0} \mid d_H(\partial(F \cap E), \partial E) \leq Mh\}.$$

Definition 4.1. For $h > 0$ the *restricted one-step discrete gradient flow* $T_{h,M}$ of the energy J with time step size h is defined by

$$T_{h,M}(E) := \arg \min_{F \in A_{r_0,M}(E)} I_h(F; E), \quad I_h(F; E) := J(F) + \frac{1}{h} \tilde{d}^2(F, E),$$

Also, the *restricted discrete gradient flow* $E_t^{h,M}$ of J for $t \in [0, \infty)$ with initial set E_0 can be defined by iterating above one-step discrete gradient flow, that is

$$E_t^{h,M} = T_{h,M}^{[t/h]}(E_0).$$

Here, T^m for $m \in \mathbb{N}$ is the m th functional power.

Next we define the notion of *restricted energy solution* as a limit of the above discrete gradient flow.

Definition 4.2. Let $w_M : \mathbb{R}^n \rightarrow \mathbb{R}$ be a *restricted energy solution* of (4.3) if for any $\epsilon > 0$ and $t > 0$, there exists $h > 0$ such that

$$\tilde{d}(\Omega_t(w_M), E_t^{h,M}) < \epsilon$$

where $E_t^{h,M}$ is the *restricted discrete gradient flow* of the energy J .

Lastly we define the notion of *energy solution* for our original problem.

Definition 4.3. (Energy Solution) $w = \chi_{\Omega_t} - \chi_{\Omega_t^c}$ is an *energy solution* of (1.4) with constant $M > 0$ if there exists $M_k \rightarrow \infty$ as k goes to infinity such that for all $t > 0$

$$d_H(\Omega_t, \Omega_t(w_{M_k})) \rightarrow 0$$

as k goes to infinity where w_{M_k} is a restricted energy solution of (4.3) with $M = M_k$.

To show the existence of the restricted energy solution, let us show several compactness property of the discrete gradient flow.

Lemma 4.1. *The volume and perimeter of the restricted discrete gradient flow $E_t^{h,M}$ is uniformly bounded in time t .*

Proof. Let us show that $|E_t^{h,M}|$ is bounded through the gradient flow. Since we have

$$J(E_{(k+1)h}^{h,M}) \leq I_h(E_{(k+1)h}^{h,M}, E_{kh}^{h,M}) \leq I_h(E_{kh}^{h,M}, E_{kh}^{h,M}) = J(E_{kh}^{h,M}),$$

for $k \in \mathbb{N}$. The energy decreases through the flow, thus we get

$$\begin{aligned} (4.5) \quad J(E_0^{h,M}) &\geq J(E_t^{h,M}) = \text{Per}(E_t^{h,M}) - \Lambda[|E_t^{h,M}|], \\ &\geq nw_n^{\frac{1}{n}} |E_t^{h,M}|^{\frac{n-1}{n}} - \Lambda[|E_t^{h,M}|], \\ (4.6) \quad &= |E_t^{h,M}|^{\frac{1}{n}} \left(nw_n^{\frac{1}{n}} - \Lambda[|E_t^{h,M}|] |E_t^{h,M}|^{\frac{1-n}{n}} \right). \end{aligned}$$

The second inequalities follows from isoperimetric inequality,

$$n^n w_n |E|^{n-1} \leq \text{Per}(E)^n.$$

for any set $E \subset \mathbb{R}^n$. Since we have

$$\lim_{|E_t^{h,M}| \rightarrow \infty} \left\{ \frac{\Lambda[|E_t^{h,M}|]}{|E_t^{h,M}|^{\frac{n-1}{n}}} \right\} = \lim_{|E_t^{h,M}| \rightarrow \infty} \left\{ \frac{\lambda[|E_t^{h,M}|]}{\frac{n-1}{n} |E_t^{h,M}|^{\frac{-1}{n}}} \right\} = \frac{n}{n-1} \lim_{|E_t^{h,M}| \rightarrow \infty} \tilde{\lambda}[|E_t^{h,M}|] < \frac{n}{n-1} \tilde{\lambda}(x_0) = nw_n^{\frac{1}{n}}$$

by Assumption A(2), the right hand side of (4.6) goes to infinity as $|E_t^{h,M}|$ goes to infinity, which implies $|E_t^{h,M}|$ is bounded.

On the other hand, the volume of $E_t^{h,M}$ is bounded from below by $|B_{r_0}|$. Thus, by (4.5) $\text{Per}(E_t^{h,M})$ is also bounded. \square

Lemma 4.2. *The energy J has a unique minimizer up to translation, which is a ball of volume s_0 .*

Proof. By first rearrangement arguments, one can show that the minimizer is a ball. By differentiating the energy $J(B_r)$ with respect to radius r , we have

$$\begin{aligned} \frac{dJ(B_r)}{dr} &= \frac{\text{Per}(B_r)}{dr} - \frac{d\Lambda[|B_r|]}{dr}, \\ &= n(n-1)w_n r^{n-2} - nw_n r^{n-1} \lambda[w_n r^n], \\ &= nw_n^{1-\frac{1}{n}} r^{n-2} ((n-1)w_n^{\frac{1}{n}} - r w_n^{\frac{1}{n}} \lambda[w_n r^n]), \\ &= nw_n^{1-\frac{1}{n}} r^{n-2} ((n-1)w_n^{\frac{1}{n}} - \tilde{\lambda}[w_n r^n]). \end{aligned}$$

By Assumption A(2), we conclude that the ball of volume s_0 become a unique minimizer of J among the radial domain. \square

Although this *pseudo-distance* does not satisfy triangle inequality, it satisfies a triangle-like inequality in $S_{r,R}$. For $F_1, \dots, F_{k+1} \in S_{r,R}$, Lemma 17 in [19] yields that

$$(4.7) \quad \frac{\tilde{d}^2(F_{k+1}, F_1)}{k} \lesssim_{r,R,N} \sum_{j=1}^k \tilde{d}^2(F_{j+1}, F_j)$$

Lemma 4.3. *The restricted discrete gradient flow $E_t^{h,M}$ in Definition 4.1 satisfies the following inequality for $0 < t_1 < t_2$:*

$$(4.8) \quad \frac{C}{t_2 - t_1} (\tilde{d}(E_{t_1}^{h,M}, E_{t_2}^{h,M}))^2 \leq J(E_{t_1}^{h,M}) - J(E_{t_2}^{h,M}).$$

Proof. Suppose that $t_1 \in [Kh, (K+1)h)$ and $t_2 \in [(K+L)h, (K+L+1)h)$ for some K and $L > 0$. By the construction of $E_t^{h,M}$ in Definition 4.1 for $k \in N$,

$$J(E_{(k-1)h}^{h,M}) - J(E_{kh}^{h,M}) \geq \frac{1}{h} \tilde{d}^2(E_{kh}^{h,M}, E_{(k-1)h}^{h,M}).$$

By adding both sides from $k = K+1$ to $k = K+L$,

$$\begin{aligned} J(E_{Kh}^{h,M}) - J(E_{(K+L)h}^{h,M}) &\geq \sum_{k=K+1}^{K+L} \frac{1}{h} \tilde{d}^2(E_{kh}^{h,M}, E_{(k-1)h}^{h,M}), \\ &\geq \frac{C}{Lh} \tilde{d}^2(E_{Kh}^{h,M}, E_{(K+L)h}^{h,M}), \end{aligned}$$

where the last inequality follows from (4.7). Therefore, we have

$$\begin{aligned} J(E_{t_1}^{h,M}) - J(E_{t_2}^{h,M}) &= J(E_{Kh}^{h,M}) - J(E_{(K+L)h}^{h,M}) \geq \frac{C}{Lh} \tilde{d}^2(E_{Kh}^{h,M}, E_{(K+L)h}^{h,M}), \\ &\geq \frac{C}{t_2 - t_1} \tilde{d}^2(E_{t_1}^{h,M}, E_{t_2}^{h,M}). \end{aligned}$$

□

Moreover, we can show that the restricted discrete gradient flow $E_t^{h,M}$ converges with respect to \tilde{d} .

Lemma 4.4. [8, Theorem 1]) Let $\{C_k\}$ be a sequence of compact star-shaped sets, which are uniformly bounded. Then, $\{C_k\}$ has a subsequence convergent in the Hausdorff metric to a compact star-shaped set.

Lemma 4.5. [19, Lemma 23, 24] Let us consider a sequence of sets $F_k \in S_{r,R}$ for $k \in \mathbb{N}$ and $F_\infty \in S_{r,R}$ for $R > r > 0$. Then, the followings are equivalent:

- (1) The pseudo-distance $\tilde{d}(F_k, F_\infty)$ goes to zero as k goes to infinity,
- (2) F_k converges to F_∞ in Hausdorff metric as k goes to infinity i.e.

$$d_H(F_k, F_\infty) = \max \left\{ \sup_{x \in F_k} \inf_{y \in F_\infty} d(x, y), \sup_{y \in F_\infty} \inf_{x \in F_k} d(x, y) \right\} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

- (3) ∂F_k converges to ∂F_∞ in Hausdorff metric as k goes to infinity.

Combing with the Lemma 4.5 and Theorem 4.4, there exists at least one restricted energy solution w_M .

Based on the proof of Proposition 3.3. in [21], we can construct the barrier property for a restricted energy solution with respect to a classical subsolution and supersolution of (2.2) with $\eta(t) = \lambda[\Omega_t(w_M)]$.

Proposition 4.6. (Barrier Property for Restricted Energy Solution) Let w_M be the restricted energy solution of (4.3) in the sense of Definition 4.2. Suppose that there exists a classical subsolution $\phi \in C^{2,1}(\mathbb{R}^n \times [0, \infty))$ of (2.2) with $\eta(t) = \lambda[\Omega_t(w_M)] - \delta$ and $\Omega_t(\phi)$ is in S_{r_1} and all $t \in [0, T]$. If

$$\Omega_0(\phi) \subset\subset \Omega_0(w_M),$$

then

$$\Omega_t(\phi) \subset\subset \Omega_t(w_M) \quad \text{for all } t \in [0, T].$$

Similarly, suppose that there exists a classical supersolution $\psi \in C^{2,1}(\mathbb{R}^n \times [0, \infty))$ of (2.2) with $\eta(t) = \lambda[\Omega_t(w_M)] + \delta$ and $\Omega_t(\psi)$ is in S_{r_1} and all $t \in [0, T]$. If we have

$$\Omega_0(w_M) \subset\subset \Omega_0(\psi),$$

then we have

$$\Omega_t(w_M) \subset\subset \Omega_t(\psi),$$

for all $t \in [0, T]$.

Proof. 1. We will prove the case $w < \psi$ at $t = 0$, parallel proof holds for the other case.

2. First, let us assume that $\Omega_t(\phi)$ touches $\Omega_t(w)$ from inside for the first time at $t = t_0$ at $x_0 \in \Omega_{t_0}(w)$. Our goal is to make a perturbation of $\Omega_t(w)$ using $\Omega_t(\phi)$, which leads to a contradiction with the gradient

flow property of w . To this end, let $\tilde{\phi}$ be a parallel translation of ϕ in the direction of normal vector at x_0 so that $\Omega_{t_0}(\tilde{\phi})$ has nonempty intersection with the complement of $\Omega_t(w)$:

$$\tilde{\phi}(x, t) := \phi \left(x - \left(e + \frac{\delta}{2}(t - t_0) \right) \vec{n}_{x_0}, t \right).$$

Then, since

$$\tilde{\phi}_t(x_0, t_0) = \phi_t(x_0, t_0) - \frac{\delta}{2} D\phi \cdot \vec{n}(x_0, t_0)$$

we have

$$(4.9) \quad \frac{\tilde{\phi}_t}{|D\tilde{\phi}|}(x_0, t_0) \leq \max \left\{ \nabla \cdot \left(\frac{D\tilde{\phi}}{|D\tilde{\phi}|} \right) (x_0, t_0) + \eta(t_0), -M \right\} - \frac{\delta}{2}.$$

Let us first assume that

$$(4.10) \quad \frac{\tilde{\phi}_t}{|D\tilde{\phi}|}(x_0, t_0) \leq \nabla \cdot \left(\frac{D\tilde{\phi}}{|D\tilde{\phi}|} \right) (x_0, t_0) + \eta(t_0) - \frac{\delta}{2}.$$

For any $\epsilon \in (0, \frac{\delta}{8})$, there exists sufficiently small $e \in (0, \frac{r_1 - r_0}{2})$ such that for $t \in [t_0 - \frac{4e}{\delta}, t_0]$, the set difference $U_t := \Omega_t(\tilde{\phi}) \setminus \Omega_t(w_M)$ satisfies $|U_t| < \epsilon$, and for all (x, t) such that

$$\sup_{t_0 - \frac{4e}{\delta} \leq s \leq t_0} d(x, U_s) < \epsilon \text{ and } t \in [t_0 - \frac{4e}{\delta}, t_0]$$

we have

$$(4.11) \quad \frac{\tilde{\phi}_t}{|D\tilde{\phi}|}(x, t) \leq \nabla \cdot \left(\frac{D\tilde{\phi}}{|D\tilde{\phi}|} \right) (x, t) - \frac{\delta}{4} + \eta(t),$$

and

$$(4.12) \quad \left| \frac{\tilde{\phi}_t}{|D\tilde{\phi}|}(x, t) - \frac{\tilde{\phi}_t}{|D\tilde{\phi}|}(x, t_0) \right| < \frac{\epsilon}{2}.$$

As (x_0, t_0) is the first touching point between $\Omega_t(\phi)$ and $\Omega_t(w_M)$, for $t \leq t_0 - \frac{2e}{\delta}$ we have

$$(4.13) \quad \Omega_t(\tilde{\phi}) \subset \Omega_t(\phi) \subset \subset \Omega_t(w_M).$$

By definition of w_M and Lemma 4.5, there exists sufficiently small $h \in (0, \frac{\epsilon}{8})$ such that the *restricted discrete gradient flow* $E_t^{h,M}$ starting from $\Omega_0(w_M)$ satisfies the following relations: for $t \leq t_0 - \frac{3e}{\delta}$, we have

$$(4.14) \quad \Omega_t(\phi) \subset E_t^{h,M},$$

and for all $t \in [t_0 - \frac{4e}{\delta}, t_0]$,

$$(4.15) \quad \left| \lambda[|E_t^{h,M}|] - \lambda[|\Omega_t(w_M)|] \right| < \epsilon, |U_t^h| < \epsilon \text{ and } d_H(U_t^h, U_t) < e$$

where U_t^h is the set difference between the zero level set of $\tilde{\phi}$ and the discrete gradient flow that is

$$U_t^h := \Omega_t(\tilde{\phi}) - E_t^{h,M}.$$

Then, there exists $k \in \mathbb{N}$ such that $\Omega_{t_0 - hk}(\phi)$ is contained in $E_{t_0 - hk}^{h,M}$ and the next step $\Omega_{t_0 - h(k-1)}(\phi)$ crosses $E_{t_0 - h(k-1)}^{h,M}$ i.e. $U_{t_0 - h(k-1)}^h$ is a nonempty set. By the relation (4.14), we have $t_0 - h(k-1) \geq t_0 - \frac{3e}{\delta}$ and thus $hk \leq \frac{4e}{\delta}$. Therefore, $x \in U_{t_0 - h(k-1)}^h$ satisfies the equations (4.11) and (4.12).

3. For simplicity, let us denote sets from the *discrete gradient scheme* $E_t^{h,M}$ by

$$F_0 = E_{t_1 - h}^{h,M}, F_h = E_{t_1}^{h,M} \text{ and } \tilde{F}_h = E_{t_1}^{h,M} \cup \Omega_{t_1}(\phi)$$

and

$$t_1 = t_0 - h(k-1)$$

Note that \tilde{F}_h is in S_{r_0} since both $E_{t_1}^h$ and $\Omega_{t_1}(\phi)$ are in S_{r_0} . Moreover, since \tilde{F}_h contains F_h ,

$$(4.16) \quad d_H(\partial(\tilde{F}_h \cap F_0), \partial F_0) \leq d_H(\partial(F_h \cap F_0), \partial F_0) \leq Mh,$$

and so \tilde{F}_h is in the admissible set. We will show that \tilde{F}_h is a minimizer of $I_h(\cdot, F_0)$ with smaller energy than for F_h , which leads to a contradiction. More precisely, we will show that

$$T_h(F_0) = \arg \min_{F \in A_{r_0, M}} I_h(F; F_0) = I_h(F_h; F_0) > I_h(\tilde{F}_h, F_0).$$

Let us write out the difference of the energies:

$$\begin{aligned} I_h(F_h; F_0) - I_h(\tilde{F}_h; F_0) &= \left(\text{Per}(F_h) - \Lambda[|F_h|] + \frac{1}{h} \tilde{d}^2(F_h, F_0) \right) \\ &\quad - \left(\text{Per}(\tilde{F}_h) - \Lambda[|\tilde{F}_h|] + \frac{1}{h} \tilde{d}^2(\tilde{F}_h, F_0) \right), \\ &= \left(\text{Per}(F_h) - \text{Per}(\tilde{F}_h) \right) + \left(-\Lambda[|F_h|] + \Lambda[|\tilde{F}_h|] \right) \\ &\quad + \left(\frac{1}{h} \tilde{d}^2(F_h, F_0) - \frac{1}{h} \tilde{d}^2(\tilde{F}_h, F_0) \right). \end{aligned}$$

Since $-\frac{D\tilde{\phi}}{|D\tilde{\phi}|}$ is outward normal velocity for $\partial\tilde{F}_h/\partial F_h$, we have

$$\begin{aligned} \text{Per}(F_h) - \text{Per}(\tilde{F}_h) &\geq \int_{\partial F_h/\partial \tilde{F}_h} d\sigma - \int_{\partial \tilde{F}_h/\partial F_h} d\sigma, \\ &\geq \int_{\partial F_h/\partial \tilde{F}_h} -\frac{D\tilde{\phi}}{|D\tilde{\phi}|}(x, t_1) \cdot \tilde{n} d\sigma - \int_{\partial \tilde{F}_h/\partial F_h} -\frac{D\tilde{\phi}}{|D\tilde{\phi}|}(x, t_1) \cdot \tilde{n} d\sigma, \\ &= \int_{\partial U_{t_1}^h} \frac{D\tilde{\phi}}{|D\tilde{\phi}|}(x, t_1) \cdot \tilde{n} d\sigma \geq \int_{U_{t_1}^h} \nabla \cdot \frac{D\tilde{\phi}}{|D\tilde{\phi}|}(x, t_1) dx. \end{aligned}$$

where \tilde{n} is outward normal vector of $\partial F_h/\partial \tilde{F}_h$ and $\partial \tilde{F}_h/\partial F_h$ at each point. The third inequality follows from the fact that outward normal of $U_{t_1}^h$ has opposite direction of outward normal of $\partial F_h/\partial \tilde{F}_h$.

Next, due to $\tilde{F}_h = F_h \cup U_{t_1}^h$ and concavity of the function Λ , we have

$$-\Lambda[|F_h|] + \Lambda[|\tilde{F}_h|] \geq \lambda[|F_h|]|U_{t_1}^h|.$$

Lastly we have

$$\begin{aligned} \frac{1}{h} \tilde{d}^2(F_h, F_0) - \frac{1}{h} \tilde{d}^2(\tilde{F}_h, F_0) &= -\frac{1}{h} \int_{U_{t_1}^h} d_{\text{signed}}(x, \partial F_0) dx, \\ &\geq -\frac{1}{h} \int_{U_{t_1}^h} d_{\text{signed}}(x, \partial \Omega_{t_1-h}(\tilde{\phi})) dx \quad (\text{by } \Omega_{t_1-h}(\tilde{\phi}) \subset F_0), \\ &\geq -\frac{1}{h} \int_{U_{t_1}^h} \left(\frac{\tilde{\phi}_t}{|D\tilde{\phi}|}(x, t_1) + \epsilon \right) h dx \quad (\text{by the relation } U_{t_1}^h \subset \Omega_{t_1}(\tilde{\phi}) \text{ and (4.12)}), \\ &\geq -\int_{U_{t_1}^h} \frac{\tilde{\phi}_t}{|D\tilde{\phi}|}(x, t_1) dx - \epsilon |U_{t_1}^h|, \end{aligned}$$

where $d_{\text{signed}}(x, \partial \Omega)$ is the signed distance function.

Putting all terms together, we have

$$\begin{aligned} I_h(F_h; F) - I_h(\tilde{F}_h; F) &\geq \int_{U_{t_1}^h} \nabla \cdot \frac{D\tilde{\phi}}{|D\tilde{\phi}|}(x, t_1) dx + \lambda[|F_h|]|U_{t_1}^h| - \int_{U_{t_1}^h} \frac{\tilde{\phi}_t}{|D\tilde{\phi}|}(x, t_1) dx - \epsilon |U_{t_1}^h|, \\ &\geq \int_{U_{t_1}^h} \frac{\delta}{4} - \lambda[|\Omega_{t_1}(w)|] dx + \lambda[|F_h|]|U_{t_1}^h| - \epsilon |U_{t_1}^h| \quad (\text{by (4.11)}), \\ (4.17) \quad &= \frac{\delta}{4} |U_{t_1}^h| + (-\lambda[|\Omega_{t_1}(w)|] + \lambda[|F_h|]) |U_{t_1}^h| - \epsilon |U_{t_1}^h|, \\ &\geq \frac{\delta}{4} |U_{t_1}^h| - \epsilon |U_{t_1}^h| - \epsilon |U_{t_1}^h| > 0 \quad (\text{by (4.15)}), \end{aligned}$$

where the last inequality follows from the fact that $\epsilon < \frac{\delta}{8}$.

4. Lastly consider the case

$$(4.18) \quad \frac{\tilde{\phi}_t}{|D\tilde{\phi}|}(x_0, t_0) \leq -M - \frac{\delta}{2}$$

in the equation (4.9). By parallel argument in step 2-3. we can choose t_1 and h sufficiently small such that for F_0 and F_h as defined in step 3, $\Omega_{t_1-h}(\phi)$ is contained in F_0 , $\Omega_{t_1}(\phi)$ crosses F_h , and ϕ satisfies (4.12) and

$$(4.19) \quad \frac{\tilde{\phi}_t}{|D\tilde{\phi}|}(x, t) \leq -M - \frac{\delta}{2}$$

for $x \in \Omega_{t_1}(\phi) \setminus F_h$ and $t \in [t_1 - h, t_1]$.

Let x^* be a point in $(\partial\tilde{F}_h) \setminus (\partial F_h)$ where $\tilde{F}_h = F_h \cup \Omega_{t_1}(\phi)$. As $\Omega_{t_1-h}(\phi)$ is contained in F_0 and Ω_t has a negative normal velocity, the point x^* is in the interior of F_0 , and thus x^* is on $\partial(\tilde{F}_h \cap F_0)$. Then, we have

$$\begin{aligned} d_H(\partial(\tilde{F}_h \cap F_0), \partial F_0) &\geq \sup_{x \in \partial(\tilde{F}_h \cap F_0)} \inf_{y \in \partial F_0} d(x, y), \\ &\geq d(x^*, \partial F_0), \\ &> d(x^*, \partial\Omega_{t_1-h}(\phi)) \quad (\text{by the relation } F_0 \subset \Omega_{t_1}(\tilde{\phi})), \\ &> (M + \frac{\delta}{2})h - \frac{\epsilon}{2}h > Mh \quad (\text{by (4.12) and (4.19)}), \end{aligned}$$

and this contradicts that \tilde{F}_h is admissible set by the equation (4.16). \square

Remark 4.7. We can see that the comparison principle does not hold as the section 3 if ϕ is not a viscosity solution of the same equation of w , but the solution of

$$\frac{\phi_t}{|D\phi|}(x, t) = \nabla \cdot \left(\frac{D\phi}{|D\phi|} \right)(x, t) + \lambda[|\Omega_t(\phi)|]$$

in the sense of Definition 2.2. This is because in this case (4.17) should be replaced by

$$I_h(F_h; F) - I_h(\tilde{F}_h; F) \geq \frac{\delta}{4}|U_{t_1}^h| + (-\lambda[|\Omega_{t_1}(\phi)|] + \lambda[|F_h|])|U_{t_1}^h| - \epsilon|U_{t_1}^h|$$

and the left hand side of the above inequalities can be negative if the volume of $\Omega_{t_1}(\phi)$ is much smaller than that of F_h .

Remark 4.8. Above comparison property does not hold, as in the section 3, if ϕ is not a viscosity solution of the same equation of w , but the solution of

$$\frac{\phi_t}{|D\phi|}(x, t) = \nabla \cdot \left(\frac{D\phi}{|D\phi|} \right)(x, t) + \lambda[|\Omega_t(\phi)|]$$

in the sense of Definition 2.2.

In the above proof, we only use the properties of the classical solution ϕ in small neighborhood of (x_0, t_0) , thus we can deduce the following localized barrier property of the energy solution.

Corollary 4.9. *Let w_M be the restricted energy solution of (4.3) in the sense of Definition 4.2. If there exists a function $\phi \in C^{2,1}(\mathbb{R}^n \times [0, \infty))$ such that $\Omega_t(\phi)$ touches above $\Omega_t(w)$ at (x_0, t_0) and*

$$(4.20) \quad x \cdot \left(-\frac{D\phi}{|D\phi|}(x) \right) \geq r > r_0,$$

then we have

$$\frac{\phi_t}{|D\phi|}(x_0, t_0) \geq \nabla \cdot \left(\frac{D\phi}{|D\phi|} \right)(x_0, t_0) + \eta(t_0).$$

Moreover, the same proof applies if we assume that the energy solution is in S_r with $r > r_0$ instead of the classical solutions, since in that case (4.20) would hold at contact points between w and the smooth barriers. Therefore we have the following corollary:

Corollary 4.10. *Let w_M be the restricted energy solution of (1.4) in the sense of Definition 4.2. If we assume that $\Omega_t(w_M)$ is in S_r for all $t \in [0, T]$ with $r > r_0$, then w_M is a viscosity solution of (2.2) with $\eta(t) = \lambda[\Omega_t(w)]$ for all $t \in [0, T]$.*

4.2. Coincidence of energy and viscosity solutions. Corollary 4.10 however does not imply that w_M coincides with the corresponding viscosity solution. To show this we will proceed instead by first proving a strict comparison principle between the energy and viscosity solutions using Corollary 4.9, and then using strict perturbations of u based on Corollary 3.8.

The standard proof for comparison principle for (2.1), such as in [15] does not apply due to the fact that we cannot perturb energy solutions out of its geometric constraint. Instead we use the doubling argument in [27] which preserve the star-shaped geometry of the level sets of the solutions.

Lemma 4.11. *Let w_M be the restricted energy solution of (4.3) in the sense of Definition 4.2. Suppose that there exists a viscosity subsolution (or supersolution) $v : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$ of (2.2) with $\eta(t) = \lambda[\Omega_t(w_M)]$ such that $\Omega_t(v)$ is in S_r , for some $r > r_0$, for all $t \in [0, T]$. If we have*

$$\Omega_0(v) \subset\subset \Omega_0(w_M) \quad (\text{ or } \Omega_0(w_M) \subset\subset \Omega_0(v)),$$

then

$$\Omega_t(v) \subset\subset \Omega_t(w_M) \quad (\text{ or } \Omega_t(w_M) \subset\subset \Omega_t(v)) \quad \text{ for all } t \in [0, T].$$

Proof. The proof is based on the doubling argument described in [13] and the second section of [27]. As in the proof of Corollary 4.9, we present the proof for (2.2) for simplicity.

First note that, by Assumption A, Lemma 4.8 as well as the fact that all admissible sets are in S_{r_0} , $\eta(t) = \lambda(\Omega_t(w))$ is in $C^{1/2}(\mathbb{R}^+)$. We will perform a space-time sup-convolution on v based on this fact. Let us choose a sufficiently small $0 < c \min[\frac{1}{2}, \frac{r-r_0}{8}]$ such that for $\delta := \min\{\frac{1}{2}, \frac{c}{4T}\}$ the function $Z : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$ defined by

$$U(x, t) := \sup_{|x-y| \leq c-2\delta t} v(y, s), \quad Z(x, t) := \sup_{|x-y|^2 + |t-s|^2 \leq c^4} U(y, s).$$

satisfies $\Omega_0(Z) \subset \Omega_0(w)$. Due to Lemma 2.4, the function U is a viscosity subsolution of $V = -H + \eta(t) - 2\delta$. Moreover, due to the Hölder continuity of η over time variable we have, from parallel arguments as in the proof of Lemma 2.4, Z is a viscosity subsolution of (the level set PDE of)

$$V = -H + \eta(t) - \delta.$$

Moreover, $\Omega_t(Z)$ is in S_{r_1-2c} because $\Omega_t(v) + y_0$ is in S_{r-c} for all point $y_0 \in \mathbb{R}^n$ such that $|y_0| \leq c$.

Let us assume that $\partial\Omega(Z)$ first touches $\partial\Omega(w_M)$ from below at (x_0, t_0) . For any $\varepsilon > 0$, let us consider functions $\tilde{Z} := \chi_{\bar{\Omega}(Z)} - 1$ and $\tilde{W} := \chi_{\Omega(w_M)}$ and

$$\Phi_\varepsilon(x, y, t) := \tilde{Z}(x, t) - \tilde{W}(y, t) - \frac{|x-y|^2}{2\varepsilon} - \frac{\varepsilon}{t_0 - t}.$$

Since the function $\tilde{Z} - \tilde{W}$ is upper semicontinuous and bounded above by zero for all $t < t_0$, the function $\Phi_\varepsilon(x, y, t)$ has a local maximizer $(x_\varepsilon, y_\varepsilon, t_\varepsilon)$ for any ε . Then, we get

$$|x_\varepsilon - y_\varepsilon| = O(\varepsilon^{1/2}).$$

Since $\tilde{Z} - \tilde{W}$ has zero at (x_0, x_0, t_0) , the local maximizer of $\Phi_\varepsilon(x, y, t)$, $(x_\varepsilon, y_\varepsilon, t_\varepsilon)$, goes to (x_0, x_0, t_0) as ε goes to zero.

By Theorem 8.3 in [13], there exists test functions $\phi^\varepsilon(x, t)$ and $\psi^\varepsilon(x, t)$ such that

$$(4.21) \quad \begin{cases} \phi^\varepsilon(x, t) = [a_\varepsilon(t - t_\varepsilon) + p_\varepsilon \cdot (x - x_\varepsilon) + \frac{1}{2}(x - x_\varepsilon)X_\varepsilon(x - x_\varepsilon)^T]_+ \geq \tilde{Z}(x, t) & \text{in } N_1^\varepsilon, \\ \psi^\varepsilon(y, t) = [b_\varepsilon(t - t_\varepsilon) + q_\varepsilon \cdot (y - y_\varepsilon) + \frac{1}{2}(y - y_\varepsilon)X_\varepsilon(y - y_\varepsilon)^T]_+ \leq \tilde{W}(y, t) & \text{in } N_2^\varepsilon, \end{cases}$$

where constants $a_\varepsilon, b_\varepsilon \in \mathbb{R}$, $p_\varepsilon, q_\varepsilon \in \mathbb{R}^n$, $X_\varepsilon, Y_\varepsilon \in S^{n \times n}$, neighborhoods N_1^ε of $(x_\varepsilon, t_\varepsilon)$ and N_2^ε of $(y_\varepsilon, t_\varepsilon)$ satisfying the inequalities:

$$(4.22) \quad \begin{cases} a_\varepsilon - b_\varepsilon & \geq 0, \\ X_\varepsilon - Y_\varepsilon & \leq \varepsilon |p_\varepsilon| I, \\ ||p_\varepsilon| - |q_\varepsilon|| & \leq \varepsilon^2 \min\{1, |p_\varepsilon|^2\}, \\ |p_\varepsilon - q_\varepsilon| & \leq \varepsilon^2 \min\{1, |p_\varepsilon|^2\}. \end{cases}$$

Since \tilde{Z} is a viscosity solution and ϕ^ε touches \tilde{Z} from above at $(x_\varepsilon, t_\varepsilon)$, we have

$$\frac{a_\varepsilon}{|p_\varepsilon|} = \frac{\phi_t^\varepsilon}{|D\phi^\varepsilon|}(x_\varepsilon, t_\varepsilon) \leq \nabla \cdot \left(\frac{D\phi^\varepsilon}{|D\phi^\varepsilon|} \right)(x_\varepsilon, t_\varepsilon) + \eta(t_\varepsilon) - \delta = \frac{1}{|p_\varepsilon|} \left(\text{trace}(X_\varepsilon) - \frac{p_\varepsilon^T X_\varepsilon p_\varepsilon}{|p_\varepsilon|^2} \right) + \eta(t_\varepsilon) - \delta.$$

By inequalities (4.22) and the ellipticity of the operator,

$$\text{trace}(X) - \frac{p^T X p}{|p|^2},$$

we can conclude that

$$\begin{aligned} \frac{b_\varepsilon}{|p_\varepsilon|} &\leq \frac{a_\varepsilon}{|p_\varepsilon|} \leq \frac{1}{|p_\varepsilon|} \left(\text{trace}(X_\varepsilon) - \frac{p_\varepsilon^T X_\varepsilon p_\varepsilon}{|p_\varepsilon|^2} \right) + \eta(t_\varepsilon), \\ &\leq \frac{1}{|p_\varepsilon|} \left(\text{trace}(Y_\varepsilon + \varepsilon |p_\varepsilon| I) - \frac{p_\varepsilon^T (Y_\varepsilon + \varepsilon |p_\varepsilon| I) p_\varepsilon}{|p_\varepsilon|^2} \right) + \eta(t_\varepsilon) - \delta. \end{aligned}$$

Also, by the strong star-shapedness of Z , we have

$$x_\varepsilon \cdot \left(-\frac{p_\varepsilon}{|p_\varepsilon|} \right) \geq r - 2c.$$

Thus, there exists sufficiently small ε_0 such that for any $\varepsilon \in (0, \varepsilon_0)$,

$$y_\varepsilon \cdot \left(-\frac{q_\varepsilon}{|q_\varepsilon|} \right) \geq \frac{r + r_0}{2} > r_0 \text{ and } \frac{b_\varepsilon}{|q_\varepsilon|} \leq \frac{1}{|q_\varepsilon|} \left(\text{trace}(Y_\varepsilon) - \frac{q_\varepsilon^T Y_\varepsilon q_\varepsilon}{|q_\varepsilon|^2} \right) + \eta(t_\varepsilon) - \frac{\delta}{2}.$$

This contradicts to the Corollary 4.9 since ψ^ε touches \tilde{W} from below at $(y_\varepsilon, t_\varepsilon)$ and the function satisfies

$$\frac{\psi_t^\varepsilon}{|D\psi^\varepsilon|} = \frac{b_\varepsilon}{q_\varepsilon} \text{ and } \nabla \cdot \frac{D\psi^\varepsilon}{|D\psi^\varepsilon|} = \frac{1}{|q_\varepsilon|} \left(\text{trace}(Y_\varepsilon) - \frac{q_\varepsilon^T Y_\varepsilon q_\varepsilon}{|q_\varepsilon|^2} \right).$$

Similarly, we can show the case that the normal velocity has lower bound by adding the restriction on the normal velocity of ϕ^ε and ψ^ε . \square

In view of above Lemma, in order to show that w_M is a viscosity solution of (2.2) with the desired $\eta(t)$, it is necessary to show that the viscosity solution u of (2.2) satisfies the star-shaped property required in the lemma. This is where we need the restricted flow with lower bound on the velocity. Recall that Theorem 3.7 guarantees the viscosity solution u of (2.2) with $\eta(t)$ coinciding with $\lambda(\Omega_t(u))$ with initial data u_0 satisfies $\Omega_t(u) \in S_{r_1}$ for all $t \geq 0$. Since we do not yet know that such u exist (we are in the process of showing this fact), we are not able to directly use this theorem.

Lemma 4.12. *Let u be a viscosity solution of (2.2) for $M > \|\eta\|_\infty$. Suppose that $\Omega_0(u)$ is in S_r and $\eta : [0, \infty) \rightarrow \mathbb{R}$ is positive. Then, for any fixed $0 < a < r$, there exists $\epsilon_0 \in (0, 1)$ such that a function $\tilde{u} : \mathbb{R}^n \times [0, \frac{a}{3M}) \rightarrow \mathbb{R}$ defined by*

$$\tilde{u}(x, t) := \inf \left\{ u \left(\frac{y}{1 + \epsilon}, t \right) \mid y \in \overline{B_{a\epsilon - 3M\epsilon t}(x)} \right\}$$

is a viscosity supersolution of (2.2) for all $0 < \epsilon < \epsilon_0$. Note that

$$\Omega_0(u) \subset \subset \Omega_0(\tilde{u}).$$

Proof. By Lemma 3.1, there exist $\epsilon_0 > 0$ such that

$$\Omega_0(u) \subset \subset \bigcap_{|z| \leq r\epsilon} [(1 + \epsilon)\Omega_0(u) + z],$$

for all $0 < \epsilon < \epsilon_0$ and $0 < a < r$. Therefore,

$$\Omega_0(u) \subset \subset \Omega_0(\tilde{u}).$$

Let us define

$$v(x, t) := u\left(\frac{x}{1 + \epsilon}, t\right)$$

for $x \in \mathbb{R}^n$ and $t \in [0, \infty)$. Then, since u is a viscosity supersolution of (2.2), v solves

$$V(x, t) \geq (1 + \epsilon) \max[-(1 + \epsilon)H(x, t) + \eta(t), -M]$$

in the viscosity sense. Now let us define

$$\tilde{u}(x, t) := \inf \{v(y, t) \mid y \in \overline{B_{a\epsilon - 3M\epsilon t}(x)}\} \text{ for } 0 < t < \frac{a}{3M}.$$

Then, one can proceed as in the proof of Lemma 2.4 to verify that \tilde{u} satisfies, in the viscosity sense,

$$V(x, t) = (1 + \epsilon) \max[-(1 + \epsilon)H(x, t) + \eta(t), -M] + 3M\epsilon$$

Since $M > \|\eta\|_\infty$ and $\eta(t)$ is positive,

$$(1 + \epsilon) \max[-(1 + \epsilon)H(x, t) + \eta(t), -M] + 3M\epsilon \geq \max\{-H(x, t) + \eta(t), -M\}$$

which implies \tilde{u} is viscosity supersolution of (2.2) for $0 \leq t < \frac{a}{3M}$. \square

Corollary 4.13. (*Short Time Strong Star-Shapedness*) Let u be a viscosity solution of (2.2). If $\Omega_0(u) \in S_r$, then, $\Omega_t(u) \in S_{r-3Mt}$ for $t \in [0, \frac{r}{3M})$.

Proof. By Lemma 4.12, there exists $\epsilon_0 \in (0, 1)$ such that

$$\tilde{u}(x, t) := \inf \left\{ u \left(\frac{y}{1 + \epsilon}, t \right) \mid y \in \overline{B_{r\epsilon-3M\epsilon t}(x)} \right\}$$

is a viscosity supersolution of (2.2) and

$$\Omega_t(u) \subset \subset \Omega_t(\tilde{u})$$

for all $0 < \epsilon < \epsilon_0$, $0 < a < r$ and $0 \leq t < \frac{r}{3M}$.

Therefore, we have

$$\Omega_t(u) \subset \subset \bigcap_{|z| \leq a\epsilon-3M\epsilon t} [(1 + \epsilon)\Omega_t(u) + z]$$

for all $0 < \epsilon < \epsilon_0$ and $0 \leq t < \frac{r}{3M}$. The above is equivalent to

$$\Omega_t(u) \in S_{r-3Mt}$$

for $t \in [0, \frac{r}{3M})$. \square

Putting Corollary 4.13 together with Lemma 4.11 in combination of Theorem 3.7, we deduce the following:

Theorem 4.14. Let w_M be a restricted energy solution. Then $\Omega_t(w_M) = \Omega_t(u)$, where u is the unique viscosity solution of (2.2) with $\eta(t) = \lambda[|\Omega_t(w_M)|]$, with initial data u_0 .

Proof. 1. Let $\eta(t)$ be as given in the theorem in the proof. The existence and uniqueness of u follows by Theorem 2.2.

2. We argue first in the small time interval $I = [0, t_0]$, where $t_0 := \frac{r_1 - r_0}{6M}$. Due to Corollary 3.8, we can make Ω_0 strictly smaller or bigger by dilation and can still make it stay in S_{r_ϵ} with $r_\epsilon = r_1 - O(\epsilon) > r_0$, where ϵ can be chosen arbitrarily small. Let us choose to make the domain strictly smaller, $\Omega_0^{\epsilon,+}$, we can apply Corollary 4.13 to ensure that the corresponding viscosity solution u^ϵ of (2.2) satisfies

$$\Omega_t(u^\epsilon) \in S_r \text{ with } r > r_0 \text{ for } 0 \leq t \leq (r_1 - r_0)/6M.$$

We can then apply Lemma 4.11 to u^ϵ and w_M to yield that $w_M \leq u^\epsilon$, and thus conclude that

$$(4.23) \quad \Omega_t(w_M) \subset \Omega_t(u^\epsilon) \text{ for } t \in I.$$

Now to send $\epsilon \rightarrow 0$, note that $\Omega_t(u^\epsilon)$ satisfies Lemma 3.10. Thus along a sequence $\epsilon = \epsilon_n \rightarrow 0$, $\Omega_t(u^\epsilon)$ converges to a domain $\Omega_t \in S_r$ uniformly with respect to d_H in the time interval I . Lemma 2.1 then yields that the corresponding level set function u for Ω_t is the unique viscosity solution of (2.2) with the initial data u_0 . From (4.23) we have

$$\Omega_t(w_M) \subset \Omega_t = \Omega_t(u) \text{ for } t \in I.$$

Similarly, using $\Omega_0^{\epsilon,-}$ instead of $\Omega_0^{\epsilon,+}$ we can conclude that $\Omega_t(u) \subset \Omega_t(w_M)$ and thus it follows that they are equal sets for the time interval I .

3. Once we know that $u = w_M$ in I , we know that $\eta(t)$ equals $\lambda(|\Omega_t(u)|)$ in I , and thus Theorem 3.7 applies and now we know that $\Omega_t(u) \in S_{r_1}$ for $t \in I$. Now we can repeat the argument at $t = t_0$ over the time interval $t_0 + I$, using the fact that $\Omega_{t_0}(u) \in S_{r_1}$. Now we can repeat above arguments to obtain that $w_M = u$ for all times. \square

4.3. Energy solution. It remains to send $M \rightarrow \infty$ to conclude our main theorem.

Theorem 4.15. *Let Ω_0 satisfy ρ -reflection. Then there exists an energy solution w of (1.4) in the sense of Definition 4.3. Moreover any energy solution w coincides with the viscosity solution of u of (2.1) with $\eta(t) = \lambda[\Omega_t(w)]$ and with initial positive set Ω_0 for all $t > 0$.*

Proof. Due to Corollary 5.3 and Theorem 4.14, we have

$$d_H(\Omega_t(w_M), \Omega_s(w_M)) \leq C|s - t|^{1/2}.$$

where C does not depend on M . Thus along a sequence $M_k \rightarrow \infty$ such that $\Omega_t(w_{M_k})$ converges with respect to d_H to Ω_t , locally uniformly in time. We conclude that $w := \chi_{\Omega_t} - \chi_{\Omega_t^c}$ of (1.4) in the sense of Definition 4.3.

Now, let us show that for any energy solution w of (1.4), the function w become the viscosity solution u of (2.1) with $\eta(t) = \lambda[\Omega_t(w)]$. By definition of the energy solution, there exists $M_k \in \mathbb{N}$ for $k > 0$ such that $M_k \rightarrow \infty$ as k goes to infinity and

$$(4.24) \quad d_H(\Omega_t(w), \Omega_t(w_{M_k})) \rightarrow 0.$$

where w_{M_k} be the restricted energy solution of (4.3) with $M = M_k$ in the sense of Definition 4.2. Since we have shown that w_{M_k} is equal to a viscosity solution of (2.2) with $\eta(t) = \lambda[\Omega_t(w_{M_k})]$, from Lemma 2.1 we can conclude that w is a viscosity solution of (2.1) with $\eta(t) = \lambda[\Omega_t(w)]$. \square

By above Theorem 4.15, u is a viscosity solution of (2.1) with $\eta(t) = \lambda[\Omega_t(w)] = \lambda[\Omega_t(u)]$. So, we can conclude that there is a solution of (1.4) in the sense of Definition 2.2.

Corollary 4.16. *There exists a solution $u : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ of (1.4) in the sense of Definition 2.2.*

5. REGULARITY AND CONVERGENCE

In this section we discuss the large time behavior of solutions for the solution of (1.4).

Let u be the energy solution obtained from Theorem 4.15. First we improve the regularity of $\Omega_t(u)$. Due to the fact that the support of u is uniformly bounded and is in S_r for all times, it follows that there exists $r_0, L_0 > 0$ such that for any point $x_0 \in \Omega_{t_0}(u)$, $\partial\Omega_{t_0}(u)$ can be represented as a Lipschitz graph in $B_r(x_0) \times [t_0 - r^2, t_0]$ with its Lipschitz constant less than L_0 . From this fact and Lemma A.1, the following lemma holds:

Lemma 5.1. *For $t \geq 1$, $\partial\Omega_t(u)$ is uniformly $C^{1,1}$. More precisely there exists $r_0, L_0 > 0$ such that for any point $x_0 \in \Omega_{t_0}(u)$, $\partial\Omega_{t_0}(u)$ can be represented as a $C^{1,1}$ graph in $B_r(x_0) \times [t_0 - r^2, t_0]$ with its $C^{1,1}$ norm less than L_0 .*

Next we show the long-time convergence of $\Omega_t(u)$ in terms of the Hausdorff distance. This is due to the fact that the ball is the unique critical point of the perimeter energy given in (4.1) among the class of $C^{1,1}$ sets.

Theorem 5.2. *$\Omega_t(u)$ uniformly converges to a ball as $t \rightarrow \infty$, modulo translation. More precisely*

$$\inf\{d_H(\Omega_t(u), B_{r^*}(x)) : x \in \overline{B_\rho(0)}\} \rightarrow 0 \text{ as } t \rightarrow \infty,$$

where B_{r^*} is the unique minimizer of the energy $J(E) = \text{Per}(E) - \Lambda[|E|]$ as discussed Remark 4.1 in section 4.

Proof. Due to the fact that $S_{r,R}$ is precompact in Hausdorff topology, along a subsequence $\Omega_t(u)$ converges to $\Omega_\infty \in S_{r,R}$. From parallel argument as in the proof of step 3, Theorem 5.1 in [19], one can verify that χ_{Ω_∞} is a viscosity solution of the prescribed mean curvature problem

$$\kappa = f(|E|),$$

where $f(|E|) = \Lambda[|E|]$ as given in (1.2). The solution of this problem, if exists, is a solution of the constant mean curvature problem. The only solution of the constant mean curvature problem, among sets in $S_{r,R}$ with $C^{1,1}$ boundaries, is smooth due to the fact that the corresponding level set PDE in the graph setting is uniformly elliptic with Lipschitz coefficients. It then follows that the only possible solution is radial, and so we conclude that $\Omega_\infty = B_{r^*}$. \square

We would still like to obtain a convergence result in higher order norms, for which it is necessary to observe several regularity properties of $\Omega_t(u)$. To this end, note that the following holds as a consequence of Lemma 5.1 and Theorem 5.2:

Lemma 5.3. *For any $\varepsilon > 0$ and $0 < \alpha < 1$ there exists T and $C > 0$ such that for any $t > T$ we have the following:*

(a) *There exists $x_t \in B_\rho(0)$ such that the outward unit normal ν_x at $\partial\Omega_t$ satisfies*

$$(5.1) \quad \left| \nu_x - \frac{(x - x_t)}{|x - x_t|} \right| \leq 2C\varepsilon \text{ on } x \in \partial\Omega_t.$$

(b)

$$(5.2) \quad \frac{|\nu_x - \nu_y|}{|x - y|^\alpha} \leq C\varepsilon^{1-\alpha}.$$

Proof. By Theorem 5.2, we can find $T > 1$ such that for any $t > T$ there exists $x_t \in B_\rho(0)$ such that

$$(5.3) \quad B_{r^*-\varepsilon}(x_t) \subset \Omega_t \subset B_{r^*+\varepsilon}(x_t).$$

Due to Lemma 5.1, the outward normal vector ν_x at $x \in \Omega_t$ satisfies

$$(5.4) \quad |\nu_x - \nu_y| \leq C|x - y|,$$

where C is independent of $t > 1$. Combining this fact with (5.3), we conclude (5.1).

From (5.4) we have

$$\frac{|\nu_x - \nu_y|}{|x - y|^\alpha} \leq Cd^{1-\alpha} \quad \text{if } |x - y| \leq d \text{ for } 0 < \alpha < 1.$$

On the other hand the same quantity is bounded by $\frac{2C\varepsilon}{d^\alpha}$ if $|x - y| \geq d$ due to (5.1). Hence choosing $d = \varepsilon$ we arrive at (5.2). \square

Lemma 5.3 states that after a finite time Ω_t gets arbitrarily close to a ball in $C^{1,\alpha}$ norm. For the volume-preserving mean curvature flow, [14] proves that when the initial domain is close to a ball in the sense of Lemma 5.3 with sufficiently small ε , it converges to a unique round ball exponentially fast. Their analysis can be also applied to our problem with minor modifications:

Theorem 5.4. *$\Omega_t(u)$ exponentially converges to a unique ball of volume s_0 whose center depending on the initial set Ω_0 , as $t \rightarrow \infty$.*

Proof. Parallel (center-manifold analysis) argument as in [14], posed in the same function space, applies here since the difference between our problem and (1.5) lies in the Lagrange multiplier $\lambda(t)$, which is a lower order term compared to the mean curvature term. \square

APPENDIX A.

Proof of Corollary 4.10. For simplicity, we consider the original case that the normal velocity is not restricted. The restricted case can be proven in the similar arguments. Let us assume that the above inequality does not hold at touching point, (x_0, t_0) ,

$$\frac{\phi_t}{|D\phi|}(x_0, t_0) < \nabla \cdot \left(\frac{D\phi}{|D\phi|} \right)(x_0, t_0) + \eta(t_0).$$

Then, there exists $\delta > 0$ such that

$$\frac{\phi_t}{|D\phi|}(x_0, t_0) \leq \nabla \cdot \left(\frac{D\phi}{|D\phi|} \right)(x_0, t_0) + \eta(t_0) - \delta.$$

At the touching point (x_0, t_0) , since w is in S_{r_1} , we have

$$x_0 \cdot \vec{n}_{x_0} = x \cdot \left(-\frac{D\phi}{|D\phi|}(x_0, t_0) \right) \geq r_1.$$

For all $\delta \in (0, \frac{r_1 - r_0}{2})$, there exist $\epsilon > 0$ such that for all $x \in B_\epsilon$ and $t \in [t_0 - \epsilon, t_0 + \epsilon]$, the inner product of the normal vector at x_0 and x is greater than δ ,

$$\left(-\frac{D\phi}{|D\phi|}(x_0, t) \right) \cdot \left(-\frac{D\phi}{|D\phi|}(x, t) \right) \geq \delta,$$

and the inequality followed by star-shapedness is preserved,

$$x \cdot \left(-\frac{D\phi}{|D\phi|}(x, t) \right) \geq r_1 - \delta.$$

There exists $e \in (0, \frac{\epsilon\delta}{4K})$ and $K > \max\{2, \epsilon\delta\}$ such that the perturbation of ϕ in the normal direction at x_0 , $\tilde{\phi}$ defined by

$$\tilde{\phi}(x, t) = \phi \left(x - \left(e + \frac{\delta}{K}(t - t_0) \right) \vec{n}_{x_0}, t \right),$$

satisfies for all $t \in [t_0 - \epsilon, t_0 + \epsilon]$,

$$(A.1) \quad U_t \subset B_\epsilon(x_0) + \left(e + \frac{\delta}{K}(t - t_0) \right) \vec{n}_{x_0},$$

where U_t is the level set difference from $\tilde{\phi}$ to w at time t that is

$$U_t = \Omega_t(\tilde{\phi}) - \Omega_t(w).$$

For all $y \in \partial U_{t_0} - \partial \Omega_{t_0}(w)$, there exist $x \in \partial \Omega_{t_0}(\phi)$ such that y can be represented by

$$y = x + e\vec{n}_{x_0}.$$

Then, for all $y \in \partial U_{t_0} - \partial \Omega_{t_0}(w)$, we have

$$\begin{aligned} y \cdot \left(-\frac{D\tilde{\phi}}{|D\tilde{\phi}|}(y, t_0) \right) &= (x + e\vec{n}_{x_0}) \cdot \left(-\frac{D\tilde{\phi}}{|D\tilde{\phi}|}(x + e\vec{n}_{x_0}, t_0) \right), \\ &= (x + e\vec{n}_{x_0}) \cdot \left(-\frac{D\phi}{|D\phi|}(x, t_0) \right) \quad (\text{by definition of } \tilde{\phi}), \\ &= x \cdot \left(-\frac{D\phi}{|D\phi|}(x, t_0) \right) + e\vec{n}_{x_0} \cdot \left(-\frac{D\phi}{|D\phi|}(x, t_0) \right) \quad (\text{by (A.1)}), \\ &\geq r_1 - \delta + e\delta > r_0. \end{aligned}$$

Similarly, for all $t \in [t_0 - \epsilon, t_0 + \epsilon]$ and all $y \in \partial U_t - \partial \Omega_t(w)$, we could show that

$$\begin{aligned} y \cdot \left(-\frac{D\tilde{\phi}}{|D\tilde{\phi}|}(y, t) \right) &= \left(x + \left(e + \frac{\delta}{K}(t - t_0) \right) \vec{n}_{x_0} \right) \cdot \left(-\frac{D\tilde{\phi}}{|D\tilde{\phi}|}(x + \left(e + \frac{\delta}{K}(t - t_0) \right) \vec{n}_{x_0}, t) \right), \\ &= (x + e\vec{n}_{x_0}) \cdot \left(-\frac{D\phi}{|D\phi|}(x, t_0) \right), \\ &= x \cdot \left(-\frac{D\phi}{|D\phi|}(x, t_0) \right) + \left(e + \frac{\delta}{K}(t - t_0) \right) \vec{n}_{x_0} \cdot \left(-\frac{D\phi}{|D\phi|}(x, t) \right) \quad (\text{by (A.1)}), \\ &\geq r_1 - \delta + \delta \left(e + \frac{\delta}{K}(t - t_0) \right) > r_1 - \delta - \delta, \\ &> r_0 \quad (\text{by } K > \epsilon\delta). \end{aligned}$$

Thus, $U_{t_0} \cup \Omega_{t_0}(w)$ is in S_{r_0} . There exists discrete energy scheme E_t starting from $\Omega_0(w)$ and $k \in \mathbb{N}$ such that $\Omega_{t_0-hk}(\phi)$ is contained in $E_{t_0-hk}^h$,

$$\Omega_{t_0-hk}(\tilde{\phi}) \subset E_{t_0-hk}^h,$$

and in the next step $\Omega_{t_0-h(k-1)}(\tilde{\phi})$ crosses $U_{t_0-h(k-1)}^h$ that is

$$U_{t_0-h(k-1)}^h := \Omega_{t_0-h(k-1)}(\tilde{\phi}) \setminus E_{t_0-h(k-1)}^h$$

is a nonempty set. Moreover, since $\Omega_t(\phi) \subset \Omega_t(w)$ for $t < t_0$, $\Omega_t(\tilde{\phi})$ touches $\Omega_t(w)$ after $e + \frac{\delta}{K}(t - t_0)$ is zero, so hk is bounded as follows,

$$hk \leq \frac{4Ke}{\delta} \leq \epsilon.$$

By the similar argument in the proof of Lemma 4.6, it can be shown that the energy from $E_{t_0-h(k-1)}^h$ to $E_{t_0-hk}^h$ is greater than the energy from $E_{t_0-h(k-1)}^h$ to $E_{t_0-h(k-1)}^h \cup U^h$,

$$I(E_{t_0-hk}^h; E_{t_0-h(k-1)}^h) \geq I(E_{t_0-h(k-1)}^h \cup U^h; E_{t_0-h(k-1)}^h).$$

This contradicts to the fact that $E_{t_0-hk}^h$ is the minimizer of

$$I_h(F; E_{t_0-h(k-1)}^h) = J(F) + \frac{1}{h} \tilde{d}^2(F, E_{t_0-h(k-1)}^h)$$

among all set in S_{r_0} . □

Lemma A.1. *Let $u(x, t)$ be a solution of*

$$(A.2) \quad \frac{\partial u}{\partial t} = \sqrt{1 + |Du|^2} \operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) + \eta(t)$$

to $Q = B_r^n(0) \times [0, r^2]$. Then for $0 < t \leq r^2$, we have the interior gradient estimate

$$(A.3) \quad |D^2 u|(0, t) \leq C(1 + \sup_Q |Du|^4) \left(\frac{1}{r^2} + \frac{1}{t} \right)$$

where the constant C depends on n and $\sup_Q |u(x, t)|$.

Proof. Based on Proposition 1.1 in [23], we have

$$\frac{\partial g_{ij}}{\partial t} = 2(\eta - H)h_{ij},$$

and

$$\frac{\partial h_{ij}}{\partial t} = \Delta h_{ij} - 2Hh_{im}h_j^m + \eta h_{im}h_j^m + |A|^2 h_{ij},$$

where $g = \{g_{ij}\}$ and $A = \{h_{ij}\}$ are the metric and the second fundamental form.

Moreover, by Corollary 1.2, we have

$$(A.4) \quad \left(\frac{\partial}{\partial t} - \Delta \right) (|A|^2) = -2|\nabla A|^2 + 2|A|^4 - 2\eta C$$

where $C = g^{ij}g^{kl}g^{mn}h_{ik}h_{lm}h_{nj}$.

Let us denote $v = \sqrt{1 + |Du|^2}$. As Lemma 3.2 in [31], the function v satisfies the equation.

$$(A.5) \quad v_t = \Delta v - |A|^2 v - \frac{2}{v} |\nabla v|^2,$$

Let us define $\phi(r) = \frac{r}{1-\delta r}$ and $g = |A|^2 \phi(v^2) = \frac{|A|^2 v^2}{1-\delta v^2}$. Then, by the similar computation of Lemma 3.2 in [31], we have

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta \right) g &= \left(\frac{\partial}{\partial t} - \Delta \right) (|A|^2) \phi(v^2) + (|A|^2) \left(\frac{\partial}{\partial t} - \Delta \right) \phi(v^2), \\ &= (-2|\nabla A|^2 + 2|A|^4 - 2\eta C) \frac{v^2}{1-\delta v^2} + 2v|A|^2 \left(-|A|^2 v - \frac{2}{v} |\nabla v|^2 \right) \times \frac{1}{(1-\delta v^2)^2}, \\ &= \frac{-2|\nabla A|^2 v^2}{1-\delta v^2} + \frac{-2\delta |A|^4 v^4}{(1-\delta v^2)^2} + \frac{-4|A|^2 |\nabla v|^2}{(1-\delta v^2)^2} - \frac{2\eta C v^2}{1-\delta v^2}, \\ &= -2|\nabla A|^2 \phi(v^2) - 2\delta g^2 + \frac{-4|A|^2 |\nabla v|^2}{(1-\delta v^2)^2} - \frac{2\eta C v^2}{1-\delta v^2}. \end{aligned}$$

The last equation follows from the fact that the second term is equal to

$$\frac{-2\delta |A|^4 v^4}{(1-\delta v^2)^2} = -2\delta g^2.$$

Since

$$\frac{1}{(1-\delta v^2)^2} = \frac{1}{1-\delta v^2} + \frac{\delta}{1-\delta v^2} \phi(v^2),$$

we get

$$\left(\frac{\partial}{\partial t} - \Delta \right) g \leq -2\delta g^2 + \frac{-2\delta |\nabla v|^2 g}{(1-\delta v^2)} - 2|\nabla A|^2 \phi(v^2) + \frac{-2|A|^2 |\nabla v|^2}{(1-\delta v^2)} + \frac{-2|A|^2 |\nabla v|^2}{(1-\delta v^2)^2} - \frac{2\eta C v^2}{1-\delta v^2}.$$

On the other hand, we have

$$\nabla g = 2A \nabla A \phi(v^2) + 2v|A|^2 \phi'(v^2) \nabla v,$$

and

$$\begin{aligned}
\phi v^{-3} \langle \nabla g, \nabla v \rangle &= \phi v^{-3} (2 \langle A \nabla A, \nabla v \rangle \phi(v^2) + v |A|^2 \phi'(v^2) |\nabla v|^2), \\
&= \phi v^{-3} (2 \langle \nabla A, A \nabla v \rangle \phi(v^2) + v |A|^2 \phi'(v^2) |\nabla v|^2), \\
&= \frac{1}{1 - \delta v^2} (2 \langle v \nabla A, A \nabla v \rangle \frac{1}{1 - \delta v^2} + 2 \frac{|A|^2 |\nabla v|^2}{(1 - \delta v^2)^2}), \\
&\leq \frac{1}{1 - \delta v^2} (\frac{v^2 |\nabla A|^2}{1 - \delta v^2} + \frac{|A|^2 |\nabla v|^2}{(1 - \delta v^2)} + 2 \frac{|A|^2 |\nabla v|^2}{(1 - \delta v^2)^2}).
\end{aligned}$$

Let us assume that $\frac{1}{1 - \delta v^2} \leq 2$. Then,

$$(\frac{\partial}{\partial t} - \Delta)g \leq -2\delta g^2 + \frac{-2\delta |\nabla v|^2 g}{(1 - \delta v^2)} - \frac{1}{2} \phi v^{-3} \langle \nabla g, \nabla v \rangle - \frac{2\eta C v^2}{1 - \delta v^2}.$$

By taking cutoff function as Lemma 3.2 in [31], we can show that

$$(g^2 + g^3)\eta t \leq ctg + g\eta$$

which implies (A.3) □

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